MATHEMATICS 30

MODULE 4

Sequences and Series

Distance Learning

Alberta EDUCATION

Mathematics 30

Module 4

SEQUENCES AND SERIES

Distance Learning

Alberta EDUCATION

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Mathematics 30
Student Module Booklet
Module 4
Sequences and Series
Alberta Distance Learning Centre
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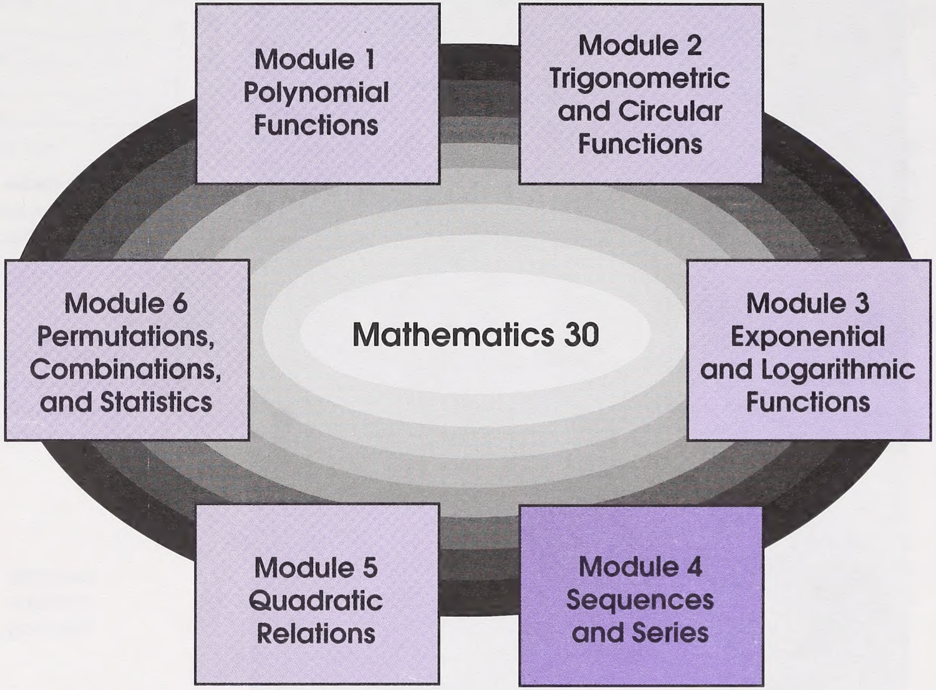
Welcome


WESTFILE INC.

Welcome to Module 4. We hope you'll enjoy your study of Sequences and Series.

Mathematics 30 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



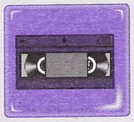
The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

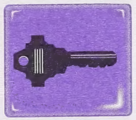
- Use your scientific calculator.

- Use computer software.

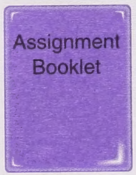
- Use the suggested answers in the Appendix to correct the activities.



- View a videocassette.

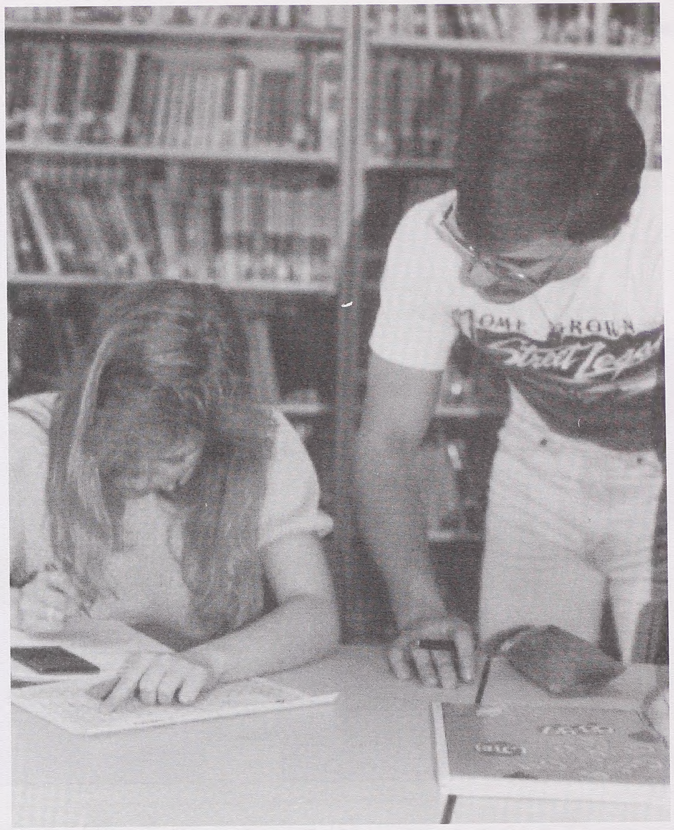


- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

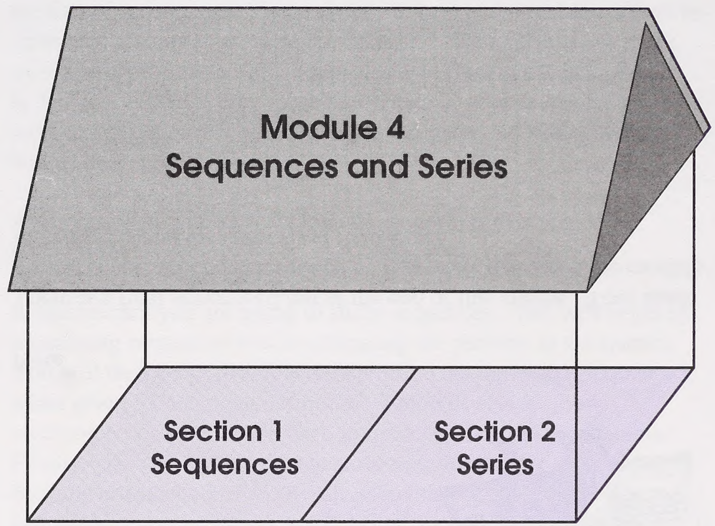
Module Overview

Sequences occur in everyday situations. Number patterns help you to remember telephone numbers. When searching for a house number, you may use part of the sequence 1, 3, 5, 7,… or part of the sequence 0, 2, 4, 6, 8,… You probably know that most house numbers ending with an even number are on the north side of an avenue or on the west side of a street. House numbers ending with an odd number are usually found on the south side of an avenue or on the east side of a street.

You probably remember mathematical problems where you are given the first few terms of a number sequence and you have to find the next few terms. For example, consider the sequence 2, 4, 6,… and the sequence 1, 4, 9, 16,… The sequences follow a distinct pattern, and the problem is to determine the pattern so that other terms of the sequence can be written.

The first twelve terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,… Did you know that the number of seeds in the spirals of a sunflower head are related to the terms in the Fibonacci sequence? The spirals on the sunflowers may contain 34 and 55 or greater numbers of seeds.

A series is obtained by adding the terms of a sequence. One main use of a series is in the calculations of compound interest investments.

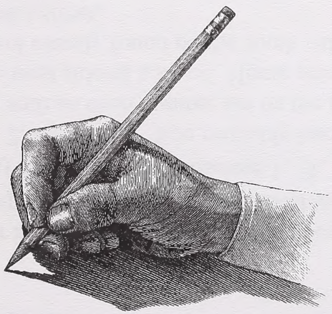


Evaluation

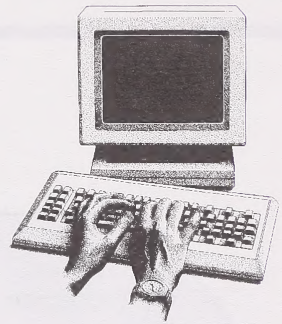
Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete two section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**40 marks**
Section 2 Assignment	**30 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.



If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Sequences


[1]

Are you a football fan? Do you watch high school football?

In a game you hear the quarterback calling some audible signals at the line of scrimmage. These signals inform and direct the offensive team, and attempt to confuse the defensive team. The quarterback uses a sequence of signals. An example is **Green, 44, 5, 2, Red, 3, 6, 5**. This sequence may mean that player 2, after receiving the ball, must hand it to player 44 on reverse, and player 44 must carry the ball through hole 6.

A calculator program or a computer program is a sequence of instructions.

In this section you are going to study sequences. You will begin by identifying sequences and investigating the patterns in a sequence. You will then be shown how to determine the terms of a sequence when given its recursive definition. You will also see how recurrence relations can be used to model real-world phenomena. Finally, you will see how the general term formula is derived for both the arithmetic and geometric sequences, and then use this formula to solve problems involving arithmetic and geometric sequences.

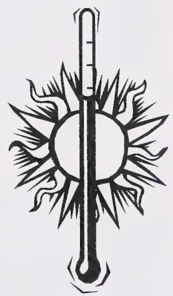
[1] The *Edmonton Journal* for the photo by Brian Gavailoff, Oct. 12, 1994, p. F3. Reprinted by permission of the *Edmonton Journal*.

Activity 1: Identifying and Determining Terms of Sequences

A **sequence** of numbers is a set of numbers arranged in a definite order. Each number in the sequence is called a **term** of the sequence.

The following high temperature readings (in °C) were recorded for a specific area on seven consecutive days:

20, 26, 30, 27, 28, 24, 23



The first number (20) represents the temperature recorded on the first day, the second number (26) represents the temperature recorded for the second day, and the last number (23) represents the temperature for the seventh day. These numbers represent a sequence, but there is no discernable pattern to this sequence.

The domain is day 1, day 2, day 3, and so on through day 7.

Domain: $\{1, 2, 3, \ldots, 7\}$

Range: $\{20, 26, 30, 27, 28, 24, 23\}$

A sequence is the range of a function whose **domain** is the natural numbers, N, or a subset of $N = \{1, 2, 3, 4, \ldots\}$.

A sequence that has no last term is called an **infinite sequence**.

Thus, $3, 7, 11, 15, \ldots$ is an infinite sequence.

The symbol "…" means **and so on** or **continued on in the same pattern**. This symbol shows that there is no last term.

A sequence that has a last term is called a **finite sequence**.

Since a finite sequence has a last term, then the number of terms in the sequence can be determined.

Thus, the sequence $2, 5, 8, 11, \ldots, 26$ is a finite sequence of nine terms.

Remember that the terms in a sequence are separated by commas.

In Section 2 you will discover that terms of a series are separated by plus or minus signs. For example, 1, 5, 9, 13, 17 is a finite sequence and $1+5+9+13+17$ is a finite series.

1. Determine if the following sequences are finite or infinite.

   a. $10, 8, 6, 4, \ldots$
   b. $2, 8, 6, 10$
   c. $-14, -11, -8, -5, \ldots, 43$
   d. $\sqrt{3}, 3, 3\sqrt{3}, 9, \ldots$
   e. $\sqrt{5}, \sqrt{5}+1, \sqrt{5}+2, \ldots, \sqrt{5}+9$
   f. $-4, -9, -14, -19, \ldots, -44$
   g. $-3, 5, -6, 10$
   h. $-30, -34, -38, -42, \ldots$
   i. $\sqrt{7}, 7, 7\sqrt{7}, 49, \ldots$
   j. $\sqrt{11}, \sqrt{11}+2, \sqrt{11}+4, \ldots$

Check your answers by turning to the Appendix.

The variable t and a subscript directly after t are used to indicate the position of a term in a sequence. For example, in the sequence $6, 10, 14, 18, 22, \ldots$, the first term is 6 and could be represented by $t_1 = 6$.

Sometimes the first term t_1 is represented by a.

The general sequence is $t_1, t_2, t_3, t_4, t_5, \ldots, t_n, \ldots$, where t_n is the nth term or general term. The fourth term of the sequence is referred to as t_4, which is 18.

It is not possible to determine the number of terms in the previous sequence since there is no last term.

The general sequence $t_1, t_2, t_3, \ldots, t_n$ is a finite sequence because t_n is the last term.

If the function is given, then the terms of a sequence can be determined. Functional notation may be used to determine the terms of a sequence. For example, $t_n = 3n-2$ or $f(n) = 3n-2$.

When $n = 4$, $t_4 = 3(4)-2 = 10$ or $f(4) = 3(4)-2 = 10$

Thus, $t_4 = f(4) = 10$. The fourth term in this sequence is 10.

Remember that n must be replaced by natural numbers $\{1, 2, 3, 4, \ldots\}$.

Look at the following examples.

Example 1

List the first five terms of the sequence determined by the function $t_n = 2n^2 - n$.

Solution

When $n = 1$,

$$\begin{aligned} t_1 &= 2(1)^2 - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

When $n = 2$,

$$\begin{aligned} t_2 &= 2(2)^2 - 2 \\ &= 2(4) - 2 \\ &= 6 \end{aligned}$$

When $n = 3$,

$$\begin{aligned} t_3 &= 2(3)^2 - 3 \\ &= 2(9) - 3 \\ &= 15 \end{aligned}$$

When $n = 4$,

$$\begin{aligned} t_4 &= 2(4)^2 - 4 \\ &= 2(16) - 4 \\ &= 28 \end{aligned}$$

When $n = 5$,

$$\begin{aligned} t_5 &= 2(5)^2 - 5 \\ &= 2(25) - 5 \\ &= 45 \end{aligned}$$

Thus, the first five terms are 1, 6, 15, 28, and 45.

Example 2

Determine the first four terms of the sequence $f : h \rightarrow 2h - 3$.

$f : h \rightarrow 2h - 3$ reads ***f* maps *h* onto $2h - 3$**.
$f(h) = 2h - 3$ reads ***f* of *h* is $2h - 3$**.

Solution

When $h = 1$,

$f : 1 \rightarrow 2(1) - 3 = -1$

When $h = 2$,

$f : 2 \rightarrow 2(2) - 3 = 1$

When $h = 3$,

$f : 3 \rightarrow 2(3) - 3 = 3$

When $h = 4$,

$f : 4 \rightarrow 2(4) - 3 = 5$

The first four terms are -1, 1, 3, and 5.

2. List the first four terms of the sequence determined by the given function.

a. $t_n = -3n^2 + 2n$ **b.** $t_n = 4n^2 - 10n$

3. Determine the first four terms of the sequence.

a. $f : h \rightarrow -4h + 7$ **b.** $f : h \rightarrow 5h - 2$

4. Determine the first four terms of a sequence that has a general term $t_n = -4n^2 + 15$. Then write this sequence as an infinite sequence.

5. Determine the first four terms of a sequence that has a general term $t_n = 8n - 2$. Then write this sequence as a finite sequence where the last term is the 12th term.

Check your answers by turning to the Appendix.

In this activity you identified sequences and terms of a sequence. You will see some patterns of sequences in the next activity.

Activity 2: Investigating Patterns in Sequences

If the first three or four terms of any finite or infinite sequence are given, then, often, you are able to find the succeeding terms and to determine the defining rule or the general term. The following examples will show how successive terms are determined by finding a rule.

Example 1

Given the sequence $4, 6, 8, \ldots$, find the next two terms.

Solution

Notice that each successive term is two more than the preceding term. Thus, add 2 to obtain each successive term.

$$\begin{aligned} t_4 &= t_3 + 2 \\ &= 8 + 2 \\ &= 10 \end{aligned} \qquad \begin{aligned} t_5 &= t_4 + 2 \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

The next two terms are 10 and 12.

Example 2

Given the sequence $2, 6, 18, 54, \ldots$, determine the next two terms.

Solution

Each term is determined by multiplying the previous term by 3.

$$\begin{aligned} t_5 &= 3t_4 \\ &= 3(54) \\ &= 162 \end{aligned} \qquad \begin{aligned} t_6 &= 3t_5 \\ &= 3(162) \\ &= 486 \end{aligned}$$

For each sequence given, complete the statements for finding each successive term; then, give the next two terms.

1. $10, 6, 2, -2, \ldots$

   Each successive term is obtained by subtracting ________ from the preceding term; or each successive term is obtained by adding ________ to the preceding term.

2. $1, 2, 4, 8, \ldots$

   Each successive term is obtained by ________ the previous term by ________.

3. $1, \frac{1}{4}, \frac{1}{16}, \ldots$

   Each successive term is obtained by multiplying the preceding term by ________; or each successive term is obtained by dividing the preceding term by ________.

4. $\sqrt{5}, 2\sqrt{5}, 4\sqrt{5}, \ldots$

   Each successive term is obtained by multiplying the preceding term by ________.

5. $3, 3\sqrt{3}, 9, \ldots$

   Multiply the previous term by ________.

Check your answers by turning to the Appendix.

The next two examples show how to determine the general term when given the first three or four terms of the sequence.

Example 3

Determine the general term for the sequence $2, 7, 12, 17, \ldots$, and use the general term to calculate the next two terms of the sequence.

Solution

Set up a table of values relating the domain values $1, 2, 3, 4$ to the range values $2, 7, 12, 17$.

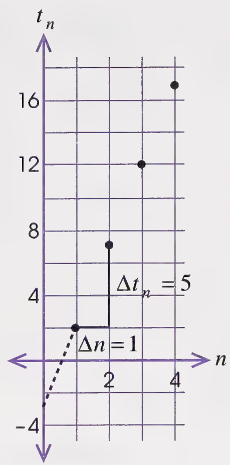
n	1	2	3	4
t_n	2	7	12	17

Remember that n represents the position of a term and t_n is the value of the nth term.

For example, when n indicates the second term, the value of t_2 is 7.

From the table of values, an increase of 1 in n produces an increase of 5 for t_n.

Notice that when ordered pairs are plotted on a graph, that the points are collinear. They could be joined by a straight line with a slope of 5. In other words, a 1 unit horizontal change in n produces a 5 unit vertical change in t_n.



The slope of a line is determined by using the coordinates of two points on the line and the following formula.

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Let $(x_1, y_1) = (2, 7)$ and $(x_2, y_2) = (3, 12)$.

Δt_n means the change in t_n.

Δn means the change in n.

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12-7}{3-2} \\ &= \frac{5}{1} \end{aligned}$$

Note that n represents the x-coordinate and t_n represents the y-coordinate.

From this information, $t_n = 5n + b$ (which is the slope-intercept form). The value of b can be determined by using any ordered pair from the table of values and substituting in $t_n = 5n + b$. Use the ordered pair $(1, 2)$.

$y = mx + b$ is the slope-intercept form, where m represents the slope of a line and b is the y-intercept.

$$\begin{aligned} 2 &= 5(1) + b \\ -3 &= b \end{aligned}$$

Thus, the general term for the given sequence is $t_n = 5n - 3$. The values of t_5 and t_6 can be determined by using the general term.

$$\begin{aligned} t_5 &= 5(5) - 3 \\ &= 22 \end{aligned}$$

$$\begin{aligned} t_6 &= 5(6) - 3 \\ &= 27 \end{aligned}$$

If the straight line joining the points is extended so that the line intersects the y-axis, then the y-intercept is -3. (See the broken line on the graph.)

Other sequences of numbers can produce functions that are not linear, but the general term can be determined by using the finite differences method.

Example 4

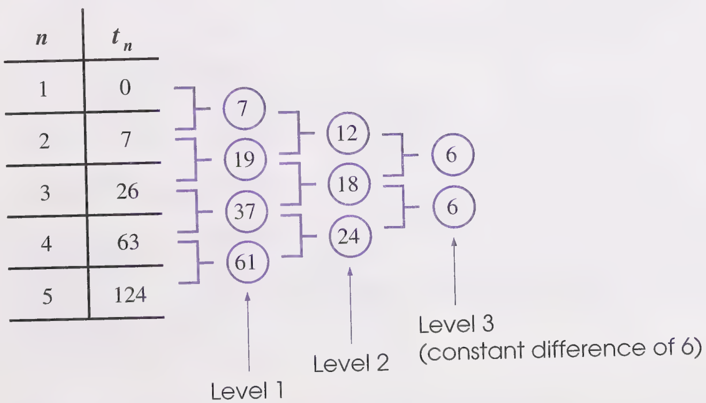
Determine the general term for the sequence 0, 7, 26, 63, 124,..., and use the general term to calculate the next two terms of the sequence.

Solution

Set up a table of values relating the domain values 1, 2, 3, 4, and 5 to the range values 0, 7, 26, 63, and 124.

n	1	2	3	4	5
t_n	0	7	26	63	124

Now subtract **adjoining** values of t_n to obtain successive differences of t_n values.



The constant difference 6 occurs at the third level; this fact implies that the general term involves a third degree polynomial. If $t_n = n^3$ is used, the ordered pairs $(1,1)$, $(2,8)$, $(3,27)$, $(4,64)$, and $(5,125)$ are obtained. Note that if 1 is subtracted from each range value, the proper values for t_n are obtained. Thus, the general term is $t_n = n^3 - 1$.

$$t_6 = 6^3 - 1 \qquad t_7 = 7^3 - 1$$
$$= 216 - 1 \qquad = 343 - 1$$
$$= 215 \qquad = 342$$

7. Determine the general term for each sequence, and use the general term to calculate the next two terms.

a. 4, 6, 8, 10,... **b.** $-5, -8, -11, -14,\ldots$

8. Determine the general term for each sequence and use the general term to calculate the next two terms.

a. 2, 5, 10, 17, 26,... **b.** $-2, 1, 6, 13, 22,\ldots$

For more practice on finding general terms of sequences, see Enrichment in this section.

9. The first term of a sequence is -10, and each successive term is four more than the previous term. Determine the general term; then calculate the value for t_{15}.

10. The first term of a sequence is 36, and each successive term is five less than the previous term. Determine the general term; then calculate the value for t_{11}.

Check your answers by turning to the Appendix.

When a sequence is identified you can determine the defining rule or the general term.

Activity 3: Using the Recursive Definition

A sequence can be defined by a formula which relates successive terms.

A **recursion formula** is used to define each term with reference to the preceding term.

You will see this in the following examples.

Example 1

Determine the first four terms of the sequence defined recursively by $t_n = t_{n-1} + 3$, where $n > 1$ and $t_1 = 4$.

Solution

To determine t_2, replace n by 2 in the formula.

$$\begin{aligned} t_n &= t_{n-1} + 3 \\ t_2 &= t_{2-1} + 3 \\ &= t_1 + 3 \\ &= 4 + 3 \qquad (\text{Given } t_1 = 4.) \\ &= 7 \end{aligned}$$

To determine t_3, replace n by 3 in the formula.

$$\begin{aligned} t_n &= t_{n-1} + 3 \\ t_3 &= t_{3-1} + 3 \\ &= t_2 + 3 \\ &= 7 + 3 \qquad (\text{From the previous calculation, } t_2 = 7.) \\ &= 10 \end{aligned}$$

To determine t_4, replace n by 4 in the formula.

$$\begin{aligned} t_n &= t_{n-1} + 3 \\ t_4 &= t_{4-1} + 3 \\ &= t_3 + 3 \\ &= 10 + 3 \qquad (\text{From the previous calculation, } t_3 = 10.) \\ &= 13 \end{aligned}$$

The first four terms are 4, 7, 10, and 13.

Example 2

Determine the first four terms of the sequence defined recursively by $t_{n+1} = t_n - 5$, where $t_1 = 3$.

Solution

To determine t_2 for the left side of the formula, n must be 1.

$$\begin{aligned} t_{n+1} &= t_n - 5 \\ t_{1+1} &= t_1 - 5 \\ t_2 &= 3 - 5 \qquad (\text{Given } t_1 = 3.) \\ &= -2 \end{aligned}$$

To determine t_3 on the left side of the formula, n must be 2.

$$\begin{aligned} t_{n+1} &= t_n - 5 \\ t_{2+1} &= t_2 - 5 \\ t_3 &= -2 - 5 \qquad (\text{From the previous calculation, } t_2 = -2.) \\ &= -7 \end{aligned}$$

To determine t_4 for the left side of the formula, n must be 3.

$$\begin{aligned} t_{n+1} &= t_n - 5 \\ t_{3+1} &= t_3 - 5 \\ t_4 &= -7 - 5 \qquad (\text{From the previous calculation, } t_3 = -7.) \\ &= -12 \end{aligned}$$

The first four terms are 3, -2, -7, and -12.

Example 3

Determine the first five terms of the sequence defined by the recursion formula $t_{n+1} = t_n + t_{n-1}$, where $n > 1$, $t_1 = 1$, and $t_2 = 1$.

Solution

In order to obtain t_3 on the left side of the formula, n must be 2.

$$\begin{aligned} t_{n+1} &= t_n + t_{n-1} \\ t_{2+1} &= t_2 + t_{2-1} \\ t_3 &= t_2 + t_1 \\ &= 1 + 1 \qquad (\text{Given } t_1 = 1 \text{ and } t_2 = 1.) \\ &= 2 \end{aligned}$$

In order to obtain t_4 on the left side of the formula, n must be 3.

$$\begin{aligned} t_{n+1} &= t_n + t_{n-1} \\ t_{3+1} &= t_3 + t_{3-1} \\ t_4 &= t_3 + t_2 \\ &= 2+1 && (t_3 \text{ was calculated in the previous step, and} \\ &= 3 && t_2 = 1 \text{ was given.)} \end{aligned}$$

In order to obtain t_5 on the left side of the formula, n must be 4.

$$\begin{aligned} t_{n+1} &= t_n + t_{n-1} \\ t_{4+1} &= t_4 + t_{4-1} \\ t_5 &= t_4 + t_3 \\ &= 3+2 && (\text{Both } t_4 \text{ and } t_3 \text{ were calculated in the} \\ &= 5 && \text{previous steps.)} \end{aligned}$$

The first five terms of the sequence are 1, 1, 2, 3, and 5.

The sequence 1, 1, 2, 3, 5,… is called the **Fibonacci** sequence. This sequence was named after Leonardo Fibonacci, an Italian mathematician.

The Fibonacci sequence begins with two ones. Any term from t_3 onwards is obtained by finding the sum of the two previous terms. For example, $t_6 = t_5 + t_4 = 5 + 3 = 8$.

1. Determine the first five terms for each recursive formula.

   a. $t_1 = 3$
   $t_n = t_{n-1} + 7$, where $n > 1$

   b. $t_1 = -4$
   $t_{n+1} = t_n - 6$

   c. $t_1 = -2$
   $t_n = t_{n-1} + 8$, where $n > 1$

   d. $t_1 = 5$
   $t_{n+1} = t_n - 10$

2. Determine the first five terms of the recursion formula $t_{n+1} = t_n + t_{n-1}$, where $n > 1$, $t_1 = 2$, and $t_2 = 3$.

3. In Example 3 of this activity, which term of the sequence has a value of 89? Solve using the Fibonacci sequence.

Check your answers by turning to the Appendix.

If you are given the first three terms of a sequence, then you should be able to determine the recursion formula. Examples 4 and 5 show you how to determine the recursion formula.

Example 4

Determine a recursion formula for the sequence $2, 7, 12, \ldots$

Solution

Note that each term is five more than the previous term.

$$t_1 = 2$$
$$t_2 = 7 \text{ or } t_2 = t_1 + 5$$
$$t_3 = 12 \text{ or } t_3 = t_2 + 5$$

Thus, the nth term t_n is five more than the t_{n-1} term.

Thus, the recursion formula is $t_n = t_{n-1} + 5$, where $n > 1$ and $t_1 = 2$ (or $t_{n+1} = t_n + 5$, where $t_1 = 2$).

Example 5

Determine a recursion formula for the sequence $2, 6, 18, \ldots$

Solution

Each term in the sequence is three times the previous term.

$$t_1 = 2$$
$$t_2 = 6 \text{ or } t_2 = 3t_1$$
$$t_3 = 18 \text{ or } t_3 = 3t_2$$

The nth term t_n is three times the t_{n-1} term. Therefore, the recursion formula is $t_n = 3t_{n-1}$, where $n > 1$ and $t_1 = 2$ or $t_{n+1} = 3t_n$, where $t_1 = 2$.

4. Determine the recursion formula for the sequence
$3, -1, -5, \ldots$

5. Determine the recursion formula for the sequence
$4, -16, 64, \ldots$

Check your answers by turning to the Appendix.

Did you see the power of sequences describing recurrence relations? Recurrence relations can be used to model real-world phenomena. The terms of the Fibonacci sequence occur frequently in nature, such as the number of seeds in the spirals of a sunflower.

Activity 4: Investigating Arithmetic Sequences

An **arithmetic sequence** is a sequence where each term is formed by adding a constant to the preceding term.

The sequence $-3, 1, 5, 9, \ldots$ is obtained by adding 4 to each term. The constant which is added to each term to produce the arithmetic sequence is called the **common difference**. In this example, the common difference is 4. The next two terms of the sequence are $9+4=13$ and $13+4=17$.

1. Determine if each sequence is arithmetic or nonarithmetic.

- **a.** $2, 4, 6, 10, \ldots$
- **b.** $-6, -8, -10, -12, \ldots$
- **c.** $-2, -4, -8, -10, -12, \ldots$
- **d.** $-8, -3, 2, 7, \ldots$
- **e.** $12, 9, 6, 3, \ldots$
- **f.** $-22, -29, -36, -43, \ldots$
- **g.** $-10, -4, 2, 8, \ldots$
- **h.** $-12, -16, -20, -23, \ldots$
- **i.** $26, 20, 14, 8, \ldots$

Check your answers by turning to the Appendix.

The **common difference** is the constant which is added to each term to produce the arithmetic sequence. You can determine the common difference of an arithmetic sequence by subtracting any term from the adjacent term that follows.

Adjacent terms are terms that are beside each other.

The arithmetic sequence $-9, -4, 1, 6, \ldots$ has a common difference of 5 since $-4-(-9)=5$, $1-(-4)=5$, or $6-1=5$.

2. Determine the common difference for each arithmetic sequence.

- **a.** $1, 5, 9, \ldots$
- **b.** $-8, -11, -14, \ldots$
- **c.** $22, 18, 14, \ldots$
- **d.** $-6, -4, -2, \ldots$
- **e.** $-6, -1, 4, \ldots$
- **f.** $7, 2, -3, \ldots$
- **g.** $30, 38, 46, \ldots$
- **h.** $-20, -5, 10, \ldots$
- **i.** $1, 1+\sqrt{2}, 1+2\sqrt{2}, \ldots$
- **j.** $3+4i, 2+2i, 1, \ldots$

Check your answers by turning to the Appendix.

The arithmetic sequence $-6, 2, 10, 18, \ldots$ has a common difference of 8. The first term t_1 is -6, and it is also represented by the variable a. The second term t_2 is 2, and it is equal to the value of the first term (-6) plus the common difference (8). The third term t_3 is 10, and it is equal to the value of the second term (2) plus the common difference (8). The fourth term t_4 is 18, and it is equal to the value of the third term (10) plus the common difference (8). The value of each term of the sequence can be shown as the sum of the first term plus a multiple of the common difference.

The first term t_1 or a is -6.

$$-6+0(8)$$

The second term t_2 is 2.

$$-6+8=-6+1(8)$$

The third term t_3 is 10.

$$-6+8+8=-6+2(8)$$

The fourth term t_4 is 18.

$$-6+8+8+8=-6+3(8)$$

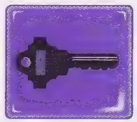
Let a represent the first term, and let d represent the common difference.

$$\therefore t_1 = a+0d$$
$$t_2 = a+1d$$
$$t_3 = a+2d$$
$$t_4 = a+3d$$

$$t_n = a+(n-1)d$$

t_n is the general term of an arithmetic sequence.

Note: The numerical coefficient of d is one less than the subscript of t.



The general term given by the formula for an arithmetic sequence is $t_n = a+(n-1)d$.

t_n is the general term of an arithmetic sequence; a is the first term; d is the common difference; and n is the number or position of the term being considered.

If any three of the values t_n, a, n, or d are known, then the fourth value can be determined.

Study the following example.

Example

For the arithmetic progression $7, 3, -1, -5, \ldots, -117$, find the number or position of the last term.

Another word for **sequence** is **progression**.

Solution

last term $t_n = -117$
first term $a = 7$
common difference $d = 3 - 7$
$= -4$

Substitute these values into the general term formula.

$$\begin{aligned} t_n &= a + (n-1)d \\ -117 &= 7 + (n-1)(-4) \\ -124 &= -4n + 4 \\ 4n &= 128 \\ n &= 32 \end{aligned}$$

Thus, -117 is the 32nd term of this arithmetic progression.

3. Determine the first four terms of the arithmetic sequence given the following values.

a. $a = 2$ and $d = 5$
b. $a = -3$ and $d = 4$
c. $a = 20$ and $d = -3$
d. $a = -16$ and $d = 6$
e. $a = -20$ and $d = -2$
f. $a = 2\sqrt{3}$ and $d = 3\sqrt{3}$

4. Determine the values for u and t in the following arithmetic sequences.

a. $4, u, 16, t, \ldots$
b. $-20, u, -36, t, \ldots$
c. $u, 23, t, -11, \ldots$

5. Find the values of the variables which represent terms of arithmetic sequences. The value of the common difference is given for each sequence.

a. $m, 2, r, t, u,$ where $d = -10$
b. $m, r, -2, t, u,$ where $d = 3$
c. $m, r, t, u, -16,$ where $d = -5$

Check your answers by turning to the Appendix.

In this activity you discovered that arithmetic sequences are such that each term is equal to the sum of the preceding term and a constant.

Activity 5: Applying the General Term Formula for Arithmetic Sequences

When given any three of the values t_n, a, n, or d, the fourth value can be determined when using the general term formula for an arithmetic sequence, $t_n = a+(n-1)d$. Study the following examples.

Example 1

Determine the 27th term of the arithmetic sequence $-30,\ -24,\ -18,\ldots$

Solution

The formula $t_n = a+(n-1)d$ is used, where t_{27} is the required value of the 27th term.

$$a=-30,\ n=27, \text{ and } d = -24-(-30)$$
$$= 6$$

$$\begin{aligned} t_n &= a+(n-1)d \\ t_{27} &= -30+(27-1)(6) \\ &= -30+(26)(6) \\ &= -30+156 \\ &= 126 \end{aligned}$$

The 27th term of the arithmetic sequence is 126.

Example 2

Determine the ninth term t_9 and the general term t_n of the arithmetic sequence $-10,\ -14,\ -18,\ldots$

Solution

first term $a=-10$

common difference $d=-14-(-10)$
$= -4$

Substitute these values into the formula $t_n = a+(n-1)d$ to determine t_9 and t_n.

$$\begin{aligned} t_n &= a+(n-1)d \\ t_9 &= -10+(9-1)(-4) \\ &= -10+(8)(-4) \\ &= -10+(-32) \\ &= -42 \end{aligned}$$

The ninth term is -42.

$$\begin{aligned} t_n &= a+(n-1)d \\ &= -10+(n-1)(-4) \\ &= -10-4n+4 \\ &= -4n-6 \end{aligned}$$

The general term is $t_n = -4n-6$.

1. **a.** Determine the tenth term of the arithmetic sequence $40, 32, 24, \ldots$

   b. Determine the 16th term of the arithmetic sequence $-20, -13, -6, \ldots$

Check your answers by turning to the Appendix.

The following shows you how to obtain the position of a term in an arithmetic sequence.

Example 3

Determine which term in the arithmetic sequence $24, 21, 18, \ldots$ is -39.

Solution

$t_n = -39$, $a = 24$, and $d = 21 - 24$
$= -3$

$$\begin{aligned} t_n &= a + (n-1)d \\ -39 &= 24 + (n-1)(-3) \\ -39 &= 24 - 3n + 3 \\ -39 &= 27 - 3n \\ 3n &= 66 \\ n &= 22 \end{aligned}$$

The 22nd term, t_{22}, has the value -39.

2. **a.** Determine which term in the arithmetic sequence $-15, -9, -3, \ldots$ is 63.

   b. Determine the number of terms in the arithmetic sequence $8, 3, -2, \ldots, -112$.

Arithmetic means are terms which lie between two given terms of an arithmetic sequence.

Now look at an example which involves arithmetic means.

Example 4

Position three arithmetic means between -10 and 14.

Solution

The given values are $t_n = 14$ and $a = -10$. When three arithmetic means are inserted between -10 and 14 there will be $2 + 3 = 5$ terms in the sequence. Thus, $n = 5$. Substitute these values into the formula $t_n = a + (n-1)d$ and determine the value of d.

$$\begin{aligned} t_n &= a + (n-1)d \\ 14 &= -10 + (5-1)d \\ 24 &= 4d \\ d &= 6 \end{aligned}$$

The arithmetic sequence is $-10, -4, 2, 8, 14$. The three arithmetic means are -4, 2, and 8.

3. a. Position four arithmetic means between -8 and 7.
   b. Position five arithmetic means between 23 and -19.

4. Determine the value of x such that $x+5$, $2+x$, and $2x+8$ form an arithmetic sequence. After calculating the value of x, determine the numerical value of the first three terms.

Check your answers by turning to the Appendix.

Study the next example.

Example 5

For an arithmetic sequence, $t_8 = -9$ and $t_{17} = -27$. Determine the common difference d, the first term a or t_1, and the general term t_n.

Solution

Substitute $t_8 = -9$ and $n = 8$ in the formula $t_n = a + (n-1)d$.

$$-9 = a + (8-1)d$$
$$-9 = a + 7d \quad (1)$$

Substitute $t_{17} = -27$ and $n = 17$ in the formula.

$$-27 = a + (17-1)d$$
$$-27 = a + 16d \quad (2)$$

Subtract (1) from (2).

$$\begin{aligned} -27 &= a + 16d \\ -9 &= a + 7d \\ \hline -18 &= 9d \\ -2 &= d \end{aligned}$$

The common difference d is -2.

To obtain the first term, substitute $d = -2$ in (1).

$$-9 = a + 7(-2)$$
$$a = 5$$

The first term a is 5.

Substitute $d = -2$ and $a = 5$ into the formula.

$$\begin{aligned} t_n &= a + (n-1)d \\ &= 5 + (n-1)(-2) \\ &= 5 - 2n + 2 \\ &= -2n + 7 \end{aligned}$$

The general term is $t_n = -2n + 7$.

5. **a.** Determine the common difference d, the first term a, and the general term t_n when $t_5 = -9$ and $t_{20} = 36$ are in an arithmetic sequence.

   b. Determine the common difference d, the first term a, and the general term t_n when $t_8 = -13$ and $t_{17} = -49$ are in an arithmetic sequence.

6. In an arithmetic sequence, the 15th term is 68 and the 30th term is 158. Determine the general term of the sequence, then find t_{21}.

Check your answers by turning to the Appendix.

The previous theory and examples in this section will now be applied to word problems. Look at the following examples.

Example 6

A new car costs $20 000. During the first year, the value of the car depreciates by $4000. Each year thereafter, the value of the car decreases by $1500. Determine the number of years when the car will be worth $5500.

Solution

After one year the car is worth $\$20\,000 - \$4000 = \$16\,000$.
After two years the car is worth $\$16\,000 - \$1500 = \$14\,500$.
After three years the car is worth $\$14\,500 - \$1500 = \$13\,000$.

The arithmetic sequence is $16\,000,\ 14\,500,\ 13\,000, \ldots$

Thus, $a = 16\,000$, $d = -1500$, and n must be determined.

$$\begin{aligned} t_n &= a + (n-1)d \\ 5500 &= 16\,000 + (n-1)(-1500) \\ 5500 &= 16\,000 - 1500n + 1500 \\ 1500n &= 12\,000 \\ n &= 8 \end{aligned}$$

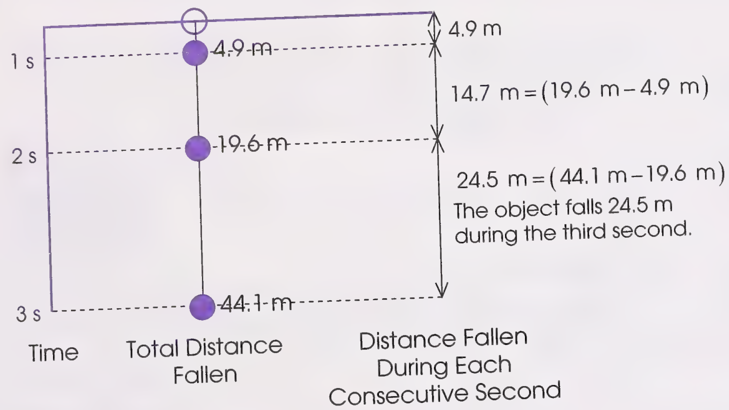
The car is worth $5500 after eight years.

A physics formula $s = \frac{1}{2}gt^2$ is used to find the distance fallen s by an object from rest under the influence of gravity. In the formula, s represents the distance (in metres), t represents the time of fall (in seconds), and g is the constant of acceleration due to gravity. The value of g is approximately $9.8\ \text{m/s}^2$.

In 1 s and no initial velocity, an object falls a total of 4.9 m.
In 2 s and no initial velocity, an object falls a total of 19.6 m.
In 3 s and no initial velocity, an object falls a total of 44.1 m.

In 1 s the object falls 4.9 m from rest.
In 2 s, the object falls a total of 19.6 m; the object falls 14.7 m during the last second of this 2 s interval.
In 3 s, the object falls a total of 44.1 m; the object falls 24.5 m during the last second of this 3 s interval.

These distances are summarized in the following diagram.



The distances fallen in each consecutive second are 4.9, 14.7, 24.5,… which form an arithmetic sequence where the common difference is 9.8. This sequence can be used to find the distance (not the total distance) fallen during any second of fall.

Example 7

A stone is dropped from a cliff. How many metres does the stone travel in the ninth second?

Solution

The arithmetic sequence is $4.9,\ 14.7,\ 24.5,\ldots$, where d is 9.8 and a is 4.9.

$$\begin{aligned} t_n &= a+(n-1)d \\ t_9 &= 4.9+(9-1)(9.8) \\ &= 4.9+78.4 \\ &= 83.3 \end{aligned}$$

The stone falls 83.3 m during the ninth second.

7. Determine the number of multiples of 5 between 52 and 303.

> A multiple of 5 is a quantity that is exactly divisible by 5.
> For example, 5, 10, 15,… are multiples of 5.

8. When money is loaned at simple interest, the money needed to pay off the loan at the end of each year is as follows:

$$M+Mi,\ M+2Mi,\ M+3Mi,\ldots,\ M+nMi$$

> M is money loaned, and i is the simple interest rate on the loan.
> Mi is the simple interest for one year on a loan of $\$M$.

a. If \$800 is loaned at 10%/a simple interest, determine the money owing at the end of each year for the first four years. Determine the amount owing at the end of n years.

b. Determine the money required to pay off a loan of \$3000 at 11%/a simple interest after eight years.

9. A highway employee has a beginning salary of \$17 000. At the end of each year, the employee obtains a raise of \$800. Determine the employee's salary at the end of eight years.

10. A car costs \$16 800. During the first year, the value of the car depreciates by \$4200. Each year thereafter the car value decreases by \$900.

    a. Determine the value of the car after five years of use.

    b. After how many years of use will the car be worth \$3600?

11. A jogger runs 300 m in the first minute; but due to fatigue, the jogger runs 20 m less in each succeeding minute. What distance does the jogger run in the ninth minute?

12. An object is dropped from a hovering helicopter and hits a water surface 12 s later. The object falls 4.9 m in the first second, 14.7 m in the next second, 24.5 m in the third second, and so on.

    a. What distance does the object fall in the 12th second?

    b. How high is the helicopter above the surface of the water (do not use the formula $s=\frac{1}{2}gt^2$)?

Check your answers by turning to the Appendix.

You can now find a term, number of terms, or the common difference by using the general term formula for arithmetic sequences.

Activity 6: Investigating Geometric Sequences

Look at the sequence 1, 4, 16, 64,… Try to determine how each term is obtained in the sequence. Each term (excluding the first term) is obtained by multiplying the previous term by 4.

A **geometric sequence** is a sequence where each term is obtained by **multiplying** the preceding term by a constant. The constant which is used to multiply each term to produce the geometric sequence is called the **common ratio**.

1. Determine if each sequence is geometric or nongeometric. For the geometric sequences, state the value of the common ratio.

    a. $2, 6, 18, 52, \ldots$

    b. $1, 3, 5, 7, \ldots$

    c. $-1, 3, -9, 27, \ldots$

    d. $-15, -5, -\frac{5}{3}, \ldots$

    e. $12, -6, 3, -\frac{3}{2}, \ldots$

    f. $5, -10, 20, 40, \ldots$

    g. $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

    h. $\frac{1}{5}, \frac{-2}{5}, \frac{4}{5}, \frac{-8}{5}, \ldots$

Check your answers by turning to the Appendix.

You can determine the common ratio r for any geometric sequence by dividing any term by the preceding term.

The geometric sequence $-6,\ 12,\ -24,\ 48, \ldots$ has a common ratio of -2 since $12 \div (-6) = -2$, $(-24) \div 12 = -2$, or $48 \div (-24) = -2$.

2. Determine the common ratio r for each geometric sequence.

a. $-10,\ -5,\ -\frac{5}{2}, \ldots$

b. $-1,\ 5,\ -25, \ldots$

c. $2,\ 5,\ 12.5, \ldots$

d. $45,\ -15,\ 5, \ldots$

e. $3,\ -12,\ 48, \ldots$

f. $80,\ -20,\ 5, \ldots$

g. $4,\ \frac{4}{5},\ \frac{4}{25}, \ldots$

h. $300,\ 200,\ 133\frac{1}{3}, \ldots$

Check your answers by turning to the Appendix.

The geometric sequence 1, 2, 4, 8,… is obtained by multiplying each term by 2. Each term in this sequence can be expressed as a power of 2.

$t_1 = 1 = 2^0$ $\qquad$ $t_2 = 2 = 2^1$

$t_3 = 4 = 2^2$ $\qquad$ $t_4 = 8 = 2^3$

$t_5 = 16 = 2^4$ $\qquad$ $t_6 = 32 = 2^5$

Note that the subscript of t is one more than the exponent of 2 in each case. Any term in a geometric sequence is represented by t_n.

t_n was also used to represent any term in an arithmetic sequence.

The formula for any term of the sequence 1, 2, 4, 8,… is $t_n = 2^{n-1}$, where n is any natural number. The formula $t_n = 2^{n-1}$ is an exponential function. The general term of a geometric sequence is an exponential function.

An exponential function contains a variable in the exponent.

The sequence $a,\ ar,\ ar^2,\ ar^3, \ldots$ is a geometric sequence where the **common ratio** is r.

Since $t_1 = a$, then $t_2 = ar$, $t_3 = ar^2$, $t_4 = ar^3$, and so on.

Therefore, the general term of a geometric sequence is $t_n = ar^{n-1}$, where a is the first term, r is the common ratio, and n is a natural number.

Note: The subscript of each term is one more than the exponent of r.

Study the following examples.

Example 1

Determine the common ratio of the following geometric sequence; then calculate the next three terms $\left(t_4, t_5, t_6\right)$.

$$\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, \ldots$$

Solution

To find the common ratio, divide any term by the previous term.

$$\left(-\frac{1}{6}\right) \div \frac{1}{3} = \left(-\frac{1}{\cancel{6}_{2}} \times \frac{\cancel{3}^{1}}{1}\right)$$
$$= -\frac{1}{2}$$

$$t_4 = \left(\frac{1}{12}\right) \times \left(-\frac{1}{2}\right) = -\frac{1}{24}$$

$$t_5 = \left(-\frac{1}{24}\right) \times \left(-\frac{1}{2}\right) = \frac{1}{48}$$

$$t_6 = \left(\frac{1}{48}\right) \times \left(-\frac{1}{2}\right) = -\frac{1}{96}$$

Example 2

Determine the common ratio of the following geometric sequence; then calculate the next three terms $\left(t_4, t_5, t_6\right)$.

$$-4, 20, -100, \ldots$$

Solution

The common ratio is $(20) \div (-4) = -5$.

$$t_4 = (-100) \times (-5) = 500$$

$$t_5 = 500 \times (-5) = -2500$$

$$t_6 = (-2500) \times (-5) = 12\,500$$

Example 3

Determine the common ratio of the following geometric sequence; then calculate the next three terms $\left(t_4, t_5, t_6\right)$.

$$3\sqrt{5}, 15, 15\sqrt{5}, \ldots$$

Solution

The common ratio is $\left(15\sqrt{5}\right) \div 15 = \sqrt{5}$.

$$t_4 = 15\sqrt{5} \times \sqrt{5} \qquad t_5 = 75 \times \sqrt{5} \qquad t_6 = 75\sqrt{5} \times \sqrt{5}$$

$$= 15(5) \qquad = 75\sqrt{5} \qquad = 75(5)$$

$$= 75 \qquad\qquad\qquad = 375$$

3. Determine the first four terms of the geometric sequence given the following values for a and r.

   a. $a = 5$ and $r = -2$

   b. $a = -2$ and $r = -\frac{1}{3}$

   c. $a = -10$ and $r = -3$

   d. $a = -60$ and $r = -\frac{3}{2}$

   e. $a = 5$ and $r = \frac{1}{5}$

   f. $a = 3$ and $r = \sqrt{7}$

4. Determine the values of the variables which represent terms of geometric sequences. The value of the common ratio is given for each sequence.

   a. $m, s, -14, t$, where $r = -7$

   b. $m, -18, s, t$, where $r = 2$

   c. $m, s, 9, t$, where $r = -\frac{1}{3}$

   d. $m, 16, s, t$, where $r = \frac{1}{4}$

Check your answers by turning to the Appendix.

Activity 7: Applying the General Term Formula for Geometric Sequences

The general term of a geometric sequence, $t_n = ar^{n-1}$, can be obtained if the values for a and r are known. The general term is used to obtain the following:

- the value of a term t_n of a geometric sequence when given a, r, and n (the position of the term)
- the position n of a term of a geometric sequence when given the value of the term t_n, a, and r

  The number of terms in a geometric sequence can be determined if the value of the last term is given along with a and r values.

- the first term a of the geometric sequence if given the values for t_n, r, and n

The value of the first term, the common ratio, and the general term can be obtained if two values of t_n are given along with their position numbers.

The following examples illustrate the use of the general term.

Example 1

Determine the general term and the seventh term of the geometric sequence $6, 18, 54, \ldots$

Solution

The first term is 6; thus, $a = 6$.
The common ratio is $18 \div 6 = 3$; thus, $r = 3$.
The general term is $t_n = ar^{n-1}$.
For this sequence, the general term is $t_n = 6(3)^{n-1}$.

$$\begin{aligned} \text{seventh term } t_7 &= ar^{7-1} \\ &= ar^6 \\ &= 6(3)^6 \quad \text{(Substitute the values for } a \text{ and } r.) \\ &= 6 \times 729 \\ &= 4374 \end{aligned}$$

Now consider this example.

Example 2

Determine the sixth term and the general term for the geometric sequence $\frac{7}{9}, \frac{7}{3}, 7, \ldots$

Solution

$$a = \frac{7}{9} \text{ and } r = \left(\frac{7}{3}\right) \div \left(\frac{7}{9}\right)$$
$$= \frac{7}{3} \times \frac{9}{7}$$
$$= 3$$

$$\begin{aligned} t_n &= ar^{n-1} \\ t_6 &= \left(\frac{7}{9}\right)(3)^{6-1} \\ &= \left(\frac{7}{9}\right)(3)^5 \\ &= \frac{7}{9}(243) \\ &= 189 \end{aligned}$$

The sixth term is 189.

$$\begin{aligned} t_n &= ar^{n-1} \\ &= \left(\frac{7}{9}\right)(3)^{n-1} \\ &= \left(\frac{7}{3^2}\right)(3)^{n-1} \\ &= 7\left(3^{-2}\right)(3)^{n-1} \\ &= 7(3)^{n-3} \end{aligned}$$

The general term is $t_n = 7(3)^{n-3}$.

Note: when $n = 1$, $t_n = 7(3)^{n-3}$
$$\begin{aligned} &= 7(3)^{1-3} \\ &= 7(3)^{-2} \\ &= \frac{7}{3^2} \\ &= \frac{7}{9} \end{aligned}$$

This agrees with the first term of the sequence.

1. Determine the first three terms given the general term of a geometric sequence.

   a. $t_n = 2\left(\frac{1}{4}\right)^{n-1}$

   b. $t_n = 3\left(-\frac{2}{3}\right)^n$

2. Determine t_5 and the general term for each geometric sequence.

   a. $-7,\ 21,\ -63, \ldots$

   b. $\frac{1}{16},\ -\frac{1}{8},\ \frac{1}{4}, \ldots$

Check your answers by turning to the Appendix.

The next example shows you how to determine the position of a term and the number of terms in a geometric sequence.

Example 3

Given the geometric sequence $-4,\ 8,\ -16, \ldots,\ 8192$, determine which term in the sequence is -1024 and find the number of terms in the sequence.

Solution

$t_n = -1024$, $a = -4$, and $r = -2$

$$\begin{aligned} \therefore t_n &= ar^{n-1} \\ -1024 &= (-4)(-2)^{n-1} && \text{(Divide both sides by } -4.) \\ 256 &= (-2)^{n-1} && \text{(Express 256 as a power of } -2.) \\ (-2)^8 &= (-2)^{n-1} \\ 8 &= n-1 \\ n &= 9 \end{aligned}$$

The exponents are equal when the bases are the same.

The ninth term has the value of -1024.

The last term in the sequence has a value of 8192. Thus, $t_n = 8192$.

$$\begin{aligned} t_n &= ar^{n-1} \\ 8192 &= (-4)(-2)^{n-1} \\ -2048 &= (-2)^{n-1} \\ (-2)^{11} &= (-2)^{n-1} \\ 11 &= n-1 \\ n &= 12 \end{aligned}$$

There are 12 terms in the geometric sequence.

3. Determine the number of terms in each geometric sequence.

   a. $-3, 6, -12, \ldots, 6144$

   b. $\frac{1}{66}, -\frac{1}{33}, \frac{2}{33}, \ldots, -\frac{512}{66}$

Check your answers by turning to the Appendix.

You can easily determine the first term if the ratio and any other term are given. However, it becomes tedious to find the first term or ratio when any two other terms of the sequence are known. The following two examples illustrate each of these types.

Example 4

Determine the first term of a geometric sequence that has a common ratio of 2 and $t_6 = 160$.

Solution

$$\begin{aligned} t_n &= ar^{n-1} \\ 160 &= a(2)^{6-1} \\ 160 &= a(2)^5 \\ 160 &= 32a \\ a &= 5 \end{aligned}$$

The first term of the geometric sequence is 5.

Example 5

In a geometric sequence, the sixth term is 972 and the eighth term is 8748. Determine a, r, and t_n for the geometric sequence.

Solution

$$\begin{aligned} t_n &= ar^{n-1} \\ t_6 &= ar^{6-1} \\ 972 &= ar^5 \end{aligned} \qquad \begin{aligned} t_n &= ar^{n-1} \\ t_8 &= ar^{8-1} \\ 8748 &= ar^7 \end{aligned}$$

Divide ar^7 by ar^5.

$$\begin{aligned} \frac{ar^7}{ar^5} &= \frac{8748}{972} \\ r^2 &= 9 \\ r &= \pm 3 \end{aligned}$$

If $r = 3$ and $t_6 = 972$, then $972 = ar^5$.

$$\begin{aligned} 972 &= a(3)^5 \\ 972 &= 243a \\ a &= 4 \end{aligned}$$

$$\begin{aligned} \therefore t_n &= ar^{n-1} \\ &= 4(3)^{n-1} \end{aligned}$$

If $r=-3$ and $t_6=972$, then $972=ar^5$.

$$972=a(-3)^5$$
$$972=-243a$$
$$a=-4$$

$$\therefore\ t_n = ar^{n-1}$$
$$= -4(-3)^{n-1}$$

4. Determine a, r, and t_n for each geometric sequence given two terms in each sequence. Then state the first three terms for each sequence.

a. $t_5=768,\ t_7=12\,288$ **b.** $t_3=7,\ t_6=\frac{7}{8}$

Check your answers by turning to the Appendix.

The **geometric means** are the terms between any two terms of a geometric sequence.

Consider this example which uses geometric means.

Example 6

Insert three geometric means between 64 and $\frac{1}{64}$.

Solution

The sequence will have a total of $3+2=5$ terms.

Thus, $t_n=\frac{1}{64}$, $a=64$, and $n=5$.

$$t_n=ar^{n-1}$$
$$\frac{1}{64}=64\left(r^{5-1}\right)$$
$$\frac{1}{64}=64r^4$$
$$\frac{1}{4096}=r^4$$
$$\pm\frac{1}{8}=r$$

If $r=\frac{1}{8}$, then the geometric means are as follows:

$$t_2=64\times\frac{1}{8} \quad t_3=8\times\frac{1}{8} \quad t_4=1\times\frac{1}{8}$$
$$=8 \quad =1 \quad =\frac{1}{8}$$

If $r=\frac{1}{8}$, the geometric sequence is $64,\ 8,\ 1,\ \frac{1}{8},\ \frac{1}{64}$.

If $r = -\frac{1}{8}$, then the geometric means are as follows:

$$t_2 = 64\left(-\frac{1}{8}\right) = -8 \qquad t_3 = (-8)\times\left(-\frac{1}{8}\right) = 1 \qquad t_4 = (1)\left(-\frac{1}{8}\right) = -\frac{1}{8}$$

If $r = -\frac{1}{8}$, the geometric sequence is $64,\ -8,\ 1,\ -\frac{1}{8},\ \frac{1}{64}$.

5. Insert three geometric means between the following numbers.

a. -5 and -80 **b.** 4 and 324

6. Three terms of a geometric sequence are $(2x+4)$, $(x-10)$, $(x-1)$. Determine the value of x; then calculate the numerical value of the three terms.

Check your answers by turning to the Appendix.

When money is loaned, there is a charge for the money loaned. This charge is called **interest (*I*)**. The sum of money loaned is called **principal (*P*)**. The length of time the money is loaned is called **time (*T*)**. **Interest rate (*R*)** is a percentage of the money loaned. Usually the interest rate (R) is interest per year unless stated otherwise. The time T is understood to be in years. The expression **per annum** means **per year**; and the word **annum** is represented by the letter **a**. (For example, 9%/a means nine percent per annum.)

The formula for calculating simple interest is $I = PRT$. The same expressions are used when money is invested. The interest paid on an investment is given at the end of equal time periods. **Simple interest** is calculated on the initial investment, so each interest payment is constant. **Compound interest** is interest paid on an investment where the previous interest has been added to the original principal investment. For example, interest is calculated on the original principal for the first time period. For the second time period, interest is calculated on the original principal plus the first period interest.

The amount of money received when a sum of money is invested for a certain time period is derived by using the following formula.

$$\boxed{A = P + I}$$

where A = the **amount** of investment,
P = **principal** deposited, and
I = **interest** earned.

Be sure you do not confuse A (amount) with P (principal).

Study the following example related to compound interest.

Example 7

One hundred dollars is deposited into a bank account that earns 8% per annum compounded annually.

Determine the amount of the investment after one year, two years, and three years.

Solution

At the end of one year, the amount A is as follows:

$$\begin{aligned} A &= P + I \\ &= 100 + 0.08(100) \\ &= 100(1 + 0.08) \\ &= 100(1.08) \\ &= 108 \end{aligned}$$

$$\begin{aligned} I &= PRT \\ &= 100\left(\frac{8}{100}\right)(1) \\ &= 0.08(100) \end{aligned}$$

The amount after one year is \$108.

The amount \$108 now becomes the principal for the second year's deposit. At the end of two years the amount is as follows:

$$\begin{aligned} A &= P + I \\ &= 108 + 0.08(108) \\ &= 100(1.08) + 0.08[100(1.08)] && \text{(Replace 108 by } 100(1.08.).) \\ &= 100(1.08)(1 + 0.08) && \text{(Remove the common factor } 100(1.08) \text{ from both terms.)} \\ &= 100(1.08)(1.08) \\ &= 100(1.08)^2 \\ &= 116.64 \end{aligned}$$

The amount after two years is \$116.64.

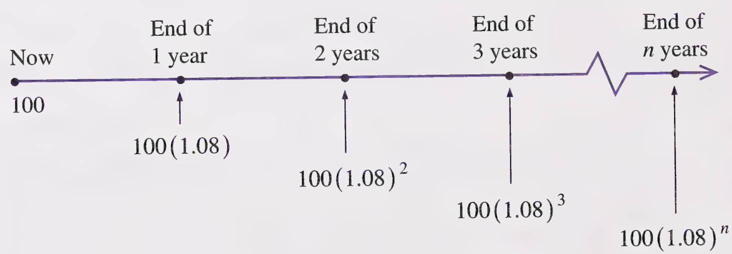
The amount \$116.64 now becomes the principal for the third year's deposit. At the end of three years, the amount is as follows:

$$\begin{aligned} A &= P + I \\ &= 116.64 + 0.08(116.64) \\ &= 100(1.08)^2 + 0.08[100(1.08)^2] && \text{(Replace 116.64 by } 100(1.08)^2.) \\ &= 100(1.08)^2[1 + 0.08] && \text{(Remove the common factor } 100(1.08)^2 \text{ from both terms.)} \\ &= 100(1.08)^3 \\ &= 125.97 \end{aligned}$$

The amount after three years is \$125.97.

Determine the general term for the geometric sequence formed from the amounts at the end of each year. Then use this general term to determine the amount at the end of seven years.

The amount at the end of each year can be shown on a time line.



The amounts at the end of each year form terms of a geometric sequence. For this sequence, $a = 100(1.08)$, $r = 1.08$, and n = number of years.

$$\begin{aligned} t_n &= ar^{n-1} \\ &= 100(1.08)(1.08)^{n-1} \\ &= 100(1.08)^n \end{aligned}$$

$$\begin{aligned} (1.08)^1(1.08)^{n-1} &= (1.08)^{1+n-1} \\ &= (1.08)^n \end{aligned}$$

The amount at the end of 7 years is $t_n = 100(1.08)^7$.

Use a calculator to find the amount.

1 0 0 × 1 • 0 8 x^y 7 =

171.3824269

The amount is $171.38 at the end of seven years.

Since calculators vary, x^y may be y^x. These keys have identical functions.

The results from **Example 7** lead to the following generalization:

$$\boxed{A = P(1+i)^n}$$

where A = the amount of investment after n periods,
P = the principal deposited,
i = the rate of interest per interest period, and
n = the number of interest periods.

When using this compound interest formula, make sure you have the proper replacement for i and n. Study the next example.

Example 8

Determine the amount of $4000 invested for nine years at 10%/annum compounded quarterly.

Quarterly means every three months.

Solution

10% per year, calculated quarterly is $\frac{10}{4} = 2.5\%$ for every quarter-year. Thus, $i = \frac{2.5}{100} = 0.025$.

Since interest is paid four times a year, there will be $4 \times 9 = 36$ interest periods in the 9 years.

$$\begin{aligned} A &= P(1+i)^n \\ &= 4000(1+0.025)^{36} \\ &= 4000(1.025)^{36} \end{aligned}$$

Use a calculator to find the amount.

4 0 0 0 × 1 . 0 2 5 x^y 3 6 =

9730.141263

The amount of the investment at the end of nine years is $9730.14.

7. Determine the value of i for each interest period; then determine n, the number of interest periods.

   a. 11% compounded annually for six years
   b. 9% compounded semiannually for seven years
   c. 12.5% compounded quarterly for five years
   d. 10% compounded monthly for three years

8. Determine the amount of the investment at the end of the time period.

   a. $800 for seven years at 11% per annum compounded annually

   b. $1600 for nine years at 10.5% per annum compounded semiannually

   c. $3400 for six years at 9.5% per annum compounded quarterly

   d. $2200 for eight years at 9% per annum compounded monthly

   e. $1500 for five years at 12% per annum compounded bimonthly

9. a. If you invest $2100 at 9%/a compounded annually, what is the amount of your investment at the end of six years?

   b. If the $2100 is invested at 9%/a compounded semiannually, what is your amount at the end of six years?

   c. If the $2100 is invested at 9%/a compounded quarterly, what is your amount at the end of six years?

   d. If the $2100 is invested at 9%/a compounded monthly, what is your amount at the end of six years?

10. On the day their child was born, the Goldsmiths invested $2000 at 11% per annum, compounded semiannually, to provide for their child's education. When the child turns seventeen, what amount of money will be saved?

Check your answers by turning to the Appendix.

If money is to be invested today to produce a future amount, then the money required today is called the present value (P_v).

In the formula $A = P(1+i)^n$, let $P = P_v$.

$$\therefore \frac{A}{(1+i)^n} = P_v$$

$$\text{Hint: } \frac{1}{(1+i)^n} = (1+i)^{-n}$$

This can be changed to the form $P_v = A(1+i)^{-n}$. P_v is the present value that must be invested today to produce the amount A at a certain interest rate i (per interest period) after n interest periods.

Look at the following example using this present value idea.

Example 9

Determine the amount of money that should be invested today at 9.5% per annum compounded semiannually in order to produce $8600 in six years.

Solution

The interest calculated semiannually is $\frac{9.5\%}{2} = 4.75\%$ for every half-year. Thus, $i = \frac{4.75}{100} = 0.0475$.

Since interest is paid two times a year, there will be $2 \times 6 = 12$ interest periods in the six years. A is $8600.

$$\begin{aligned} \therefore P_v &= A(1+i)^{-n} \\ &= 8600(1+0.0475)^{-12} \\ &= 8600(1.0475)^{-12} \end{aligned}$$

Use a calculator to find P_v.

8 6 0 0 × 1 . 0 4 7 5 x^y 1 2 +/– =

4927.765942

The amount $4927.77 should be invested today in order to have $8600 in six years.

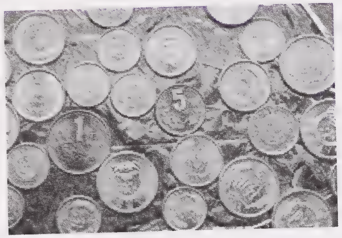
11. If Tamika needs $3600 in five years to further her education, what amount of money should she invest today at 11.5%/a compounded semiannually?

Check your answers by turning to the Appendix.

You have seen how geometric sequences apply to the mathematics of investment. Similarly, bacterial growth, half-life of chemicals, and population growth are some of the other areas that can be represented by a geometric sequence.

12. A certain bacteria divides into four every hour. How many bacteria will there be after 8 h?

13. Mr. Jacobs, a coin collector, triples his collection every five years. After twenty years, he has a collection of 3240 coins. How many coins did Mr. Jacobs begin with?



14. The zebra population in an African country is 3600. It is expected that this population will increase by 4%/a for five years. Write an expression to show the expected population after one year. Use this expression to find the population after four years.

Check your answers by turning to the Appendix.

You can find the term, number of terms, or the common ratio using the general term formula of geometric sequences. You have also seen the many applications of this formula to real-world problems.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The word **sequence** occurs quite often in everyday language. You have a sequence of events, a sequence of time, and so on. Can you think of some areas that use sequences?

If you look at the arrangement of seeds in a sunflower, you will see clockwise and counterclockwise spiral patterns with a definite number of seeds for each pattern.

The phone book contains a sequence of numbers.

A small planet, called Ceres, was discovered using a specific number sequence by Carl Friedrich Gauss (1777 to 1855 A.D).

1. Fill in the blanks.

   a. A __________ of numbers is a set of numbers arranged in a definite order.

   b. A sequence is a function whose __________ is the natural numbers and whose __________ is the set of real or complex numbers.

   c. A sequence that has no last term is called an __________ sequence.

d. A sequence that has a last term is called a __________ sequence.

e. When a sequence of numbers is given, then this sequence is usually the __________ of the sequence.

f. The __________ in a sequence are separated by commas.

2. Determine the first five terms of the sequence for the function $t_n = -n^2 + 3n$.

3. Determine the first four terms of a sequence that has a general term $t_n = -4n + 9$. Then write this sequence as a finite sequence where the last term is the seventeenth term.

4. Determine the general term for the sequence $7, 10, 15, 22, \ldots,$ and use the general term to calculate the next two terms of the sequence.

5. The first term of a sequence is 20 and each successive term is six less than the previous term. Determine the general term; then calculate the value for t_{16}.

6. Determine the first four terms of the sequence defined recursively by $t_{n+1} = t_n + 8$ if $t_1 = 4$.

7. Determine a recursion formula for the sequence $-4, -10, -16, \ldots$

Check your answers by turning to the Appendix.

You may find it helpful to review the concept of sequences by viewing the video titled *Arithmetic Sequences and Series* from the *Catch 30* series, ACCESS Network. This video is available from the Learning Resources Distributing Centre. (**Note:** You may view the concept of series after studying Section 2.)

A sequence of numbers $t_1, t_2, t_3, \ldots, t_n$ is called an **arithmetic sequence** only if that sequence can be transformed into the general arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$, where a represents t_1 and d represents the common difference.

In an arithmetic sequence, each succeeding term is formed from the preceding term by adding a constant to it.

An arithmetic sequence of terms may contain a variable. The numerical value of the term can be calculated by finding the common difference. Study the following example.

Example 1

Find the positive value of m such that $\frac{1}{m}$, 1, and $\frac{6}{m+2}$ form an arithmetic sequence. Then determine the value of the three terms.

Solution

Let d_1 be the difference between the first two terms, and let d_2 be the difference between terms two and three.

$$\begin{aligned} d_1 &= t_2 - t_1 \\ &= (1) - \left(\frac{1}{m}\right) \end{aligned}$$

$$\begin{aligned} d_2 &= t_3 - t_2 \\ &= \left(\frac{6}{m+2}\right) - (1) \end{aligned}$$

The common difference between the terms are the same.

$$\begin{aligned} \therefore d_1 &= d_2 \\ (1) - \left(\frac{1}{m}\right) &= \left(\frac{6}{m+2}\right) - (1) \\ m(m+2) - (m+2) &= 6m - m(m+2) \\ m^2 + 2m - m - 2 &= 6m - m^2 - 2m \\ 2m^2 - 3m - 2 &= 0 \\ (2m+1)(m-2) &= 0 \\ m &= 2 \qquad (\text{Since } m \text{ is positive.}) \end{aligned}$$

The first term is $\frac{1}{m} = \frac{1}{2}$. The second term is 1. The third term is $\frac{6}{m+2} = \frac{6}{2+2} = \frac{3}{2}$.

8. State the values of a and d for each arithmetic sequence.

a. $-5, -9, -13, \ldots$ **b.** $19, 16, 13, \ldots$

c. $-8, -3, 2, \ldots$ **d.** $16, 6, -4, \ldots$

9. For the arithmetic sequence $20, 5, -10, \ldots, -310$, determine the number or position of the last term.

10. Determine the 45th term of the arithmetic sequence $-64, -56, -48, \ldots$

11. Determine t_{18} and the general term t_n of the arithmetic sequence $56, 47, 38, \ldots$

12. Insert four arithmetic means between -32 and 28.

13. For an arithmetic sequence, $t_3 = -12$ and $t_{14} = 54$. Determine the common difference, the first term, and the general term.

14. Determine the value of x such that $2x-2$, $2x+5$, and $x+5$ form an arithmetic sequence. After calculating the value of x, determine the value of the three terms.

Check your answers by turning to the Appendix.

You may find it helpful to review the concept of sequences by viewing the video titled *Geometric Sequences and Series* from the *Catch 30* series, ACCESS Network. This video is available from the Learning Resources Distributing Centre. (**Note:** You may view the concept of series after studying Section 2).

The terms between the first and last terms of a geometric sequence are called **geometric means**. You may have one or more means. To find one mean, you can use the ratio method for fractions.

Look at the following example.

Example 2

Determine the geometric mean between 3 and 48.

Solution

Let x be the middle term.

The sequence is $3,\ x,\ 48$.

Since you have a common ratio,

$$\begin{aligned} \frac{x}{3} &= \frac{48}{x} \\ (x)(x) &= (48)(3) \\ x^2 &= 144 \\ x &= \pm 12 \end{aligned}$$

The geometric mean is either 12 or -12.

15. Determine the geometric mean between $\frac{1}{2}$ and 32.

16. In a geometric sequence, the eleventh term $\left(t_{11}\right)$ is 1536 and the ninth term $\left(t_9\right)$ is 96. Determine the common ratio.

17. Determine the positive geometric mean between $-\frac{1}{3}$ and -243.

18. Three brothers shared some money which formed a geometric sequence. If the oldest and youngest received \$180 and \$45 respectively, how much did the remaining one get?

Check your answers by turning to the Appendix.

Enrichment

The following examples show how the general term is determined for more difficult sequence patterns.

Example 1

Determine the general term for the sequence $2,\ 4,\ 8,\ 16, \ldots$

Solution

Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	2	4	8	16

Each term is obtained by multiplying the previous term by 2. When terms are obtained by multiplying by a constant, this usually means the sequence is represented by an exponential function. Since the constant is 2, express each term of the sequence as a power of 2.

$$2 = 2^1,\ 4 = 2^2,\ 8 = 2^3,\ 16 = 2^4$$

n	1	2	3	4
t_n	2^1	2^2	2^3	2^4

There is an increase of one in each exponent when n increases by one. Try the exponent $+1n$ since the exponent increases by one.

For the general term, try $t_n = 2^n$.

When $n = 3$, $t_n = 2^3$ (from the table of values). Thus, $2^3 = 2^3$ and the general term is $t_n = 2^n$.

Example 2

Determine the general term for the sequence $64, 32, 16, 8, \ldots$

Solution

Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	64	32	16	8

Each term is obtained by multiplying the previous term by $\frac{1}{2}$. Since the constant is $\frac{1}{2}$ or 2^{-1}, try to express each term of the sequence as a power of 2.

$$64 = 2^6,\ 32 = 2^5,\ 16 = 2^4,\ 8 = 2^3$$

n	1	2	3	4
t_n	2^6	2^5	2^4	2^3

There is a decrease of one in each exponent when n increases by one. Try the exponent $-1n$ since the exponent decreases by one.

For the general term, try $t_n = 2^{-n}$.

When $n = 2$, $t_n = 2^5$ (from the table of values). Does $2^5 = 2^{-2}$? No, but if 7 is added to the exponent on the right side, then the two sides will be equal. Try $t_n = 2^{-n+7}$ which satisfies the given values.

Thus, the general term is $t_n = 2^{-n+7}$.

Example 3

Determine the general term for the sequence 12, 24, 48, 96,…

Solution

Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	12	24	48	96

Each term is obtained by multiplying the previous term by 2. Each term of the sequence cannot be expressed as a power of 2, but it may be possible to express each term as a multiple of a power of 2, that is $k2^n$. Set up a table of values showing n, t_n, and 2^n.

n	1	2	3	4
t_n	12	24	48	96
2^n	2	4	8	16

Examine the t_n and 2^n values. The t_n values can be obtained by multiplying the 2^n values by 6.

Thus, try the general term $t_n = 6(2)^n$.

When $n = 3$, $t_n = 6(2)^3$

$= 48$ (agrees with the given value)

Thus, the general term is $t_n = 6(2)^n$.

Example 4

Determine the general term for the sequence 135, 45, 15, 5,…

Solution

Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	135	45	15	5

Each term is obtained by multiplying the previous term by $\frac{1}{3}$. Each term of the sequence cannot be expressed as a power of 3, but it may be possible to express each term as a multiple of a power of 3. Set up a table of values showing n, t_n, and 3^{-n}.

Note that 3^{-n} is used instead of 3^n because the t_n values are decreasing as n increases.

n	1	2	3	4
t_n	135	45	15	5
3^{-n}	3^{-1}	3^{-2}	3^{-3}	3^{-4}

Try the general term $t_n = k\left(3^{-n}\right)$. When $n = 1$, $t_n = 135$.

$$\begin{aligned} \therefore\ 135 &= k\left(3^{-1}\right) \\ 135 &= k\left(\frac{1}{3}\right) \\ k &= 405 \end{aligned}$$

Thus, $t_n = 405\left(3^{-n}\right)$ or $\frac{405}{3^n}$.

Check

Check if this formula satisfies the other values of n and t_n.

$$\begin{aligned} \text{If } n = 3,\ t_3 &= \frac{405}{3^3} \\ &= \frac{405}{27} \\ &= 15 \end{aligned}$$

This agrees with the third term in the sequence. Thus, the general term is $t_n = \frac{405}{3^n}$.

Now do any or all of the following questions.

1. Determine the general term for the sequence $2, 6, 18, 54, \ldots$
2. Determine the general term for the sequence $136, 68, 34, 17, \ldots$
3. Determine the general term for the sequence $5, 10, 20, 40, \ldots$
4. Determine the general term for the sequence $108, 36, 12, 4, \ldots$

Check your answers by turning to the Appendix.

For any arithmetic sequence, you can determine the general term t_n. The numerical value of a term can then be determined from this formula. The next example shows you a general term formula.

Example 5

Determine the general term and the ninth term for the arithmetic sequence $-1, 4, 9, 14$.

Solution

The common difference d is $4 - (-1) = 5$.

$$\begin{aligned} t_n &= a + (n-1)d \\ &= -1 + (n-1)(5) \\ &= -1 + 5n - 5 \\ &= 5n - 6 \end{aligned}$$

The general term is $t_n = 5n - 6$.

$$t_9 = 5(9) - 6$$
$$= 39$$

The ninth term is 39.

Most of these problems will provide you with more practice beyond that already shown in this section.

5. Determine the general term t_n and the seventh term for the arithmetic sequence $\frac{3}{y}, \frac{6}{y}, \frac{9}{y}, \ldots$

6. Determine the general term t_n and the number of terms in the arithmetic sequence $4u - 3t,\ 5u - t,\ 6u + t, \ldots,\ 13u + 15t$.

7. In an arithmetic sequence, t_1 and t_3 have a sum of 32. Also t_2 and t_4 have a sum of 24. Calculate the common difference and the first four terms.

8. Suppose $r,\ 10,\ t, \ldots$ form an arithmetic sequence. Determine r and t if $r^2 + t^2 = 298$.

Check your answers by turning to the Appendix.

To calculate **time** in the compound interest formula, you have to use logarithms. The following example illustrates this.

Example 6

If \$1000 is deposited at 8%/a compounded semiannually, what is the time required for this \$1000 to double in value?

Solution

8% per year calculated semiannually is $\frac{8}{2} = 4\%$ for every half year.

$$\therefore\ i = \frac{4}{100}$$
$$= 0.04$$

The variable n represents the number of six-month intervals.

A is \$2000, and P is \$1000.

$$A = P(1+i)^n$$
$$2000 = 1000(1+0.04)^n$$
$$2000 = 1000(1.04)^n \qquad \text{(Divide both sides by 1000.)}$$
$$2 = (1.04)^n$$

Use logarithms to solve for n.

$$\log 2 = \log(1.04)^n$$
$$\log 2 = n\log(1.04) \qquad \text{(Power Law)}$$
$$\frac{\log 2}{\log 1.04} = n$$

Use a calculator to find n.

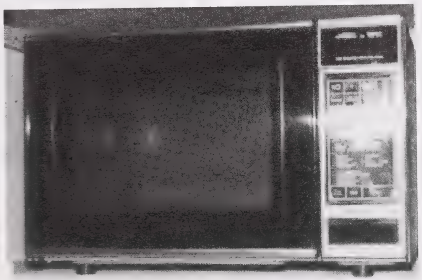
2 log ÷ 1 . 0 4 log =

17.67298769

Thus, $n \doteq 17.67$ six-month intervals.

The number of interest periods is rounded to the next highest period. Thus, the money will double in 18 six-month intervals or nine years.

9. A microwave costs \$475. Sam invests \$400 at 12% per annum compounded every two months. How long must Sam wait before he has enough money to buy the microwave?



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Check your answers by turning to the Appendix.

To determine the terms in a geometric sequence for some problems will require the use of a system of equations and the ratio method.

Study the example shown.

Example 7

In a geometric sequence, the sum of the third and fourth terms is 240. Also, the sum of the seventh and eighth terms is 15. Determine the first eight terms of this sequence.

Solution

$$t_n = ar^{n-1} \qquad t_n = ar^{n-1} \qquad t_n = ar^{n-1} \qquad t_n = ar^{n-1}$$
$$t_3 = ar^2 \qquad t_4 = ar^3 \qquad t_7 = ar^6 \qquad t_8 = ar^7$$

$$t_3 + t_4 = 240 \qquad\qquad t_7 + t_8 = 15$$
$$ar^2 + ar^3 = 240 \qquad\qquad ar^6 + ar^7 = 15$$
$$ar^2(1+r) = 240 \quad (1) \qquad\qquad ar^6(1+r) = 15 \quad (2)$$

Divide equation (2) by (1).

$$\frac{ar^6(1+r)}{ar^2(1+r)} = \frac{15}{240}$$
$$r^4 = \frac{1}{16}$$
$$r = \pm\frac{1}{2}$$

Substitute $r = \frac{1}{2}$ in ①.

$$a\left(\frac{1}{2}\right)^2\left(1+\frac{1}{2}\right) = 240$$
$$a\left(\frac{1}{4}\right)\left(\frac{3}{2}\right) = 240$$
$$\frac{3a}{8} = 240$$
$$a = 640$$

If $r = \frac{1}{2}$, the first eight terms are 640, 320, 160, 80, 40, 20, 10, and 5.

Substitute $r = -\frac{1}{2}$ in ①.

$$a\left(-\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right) = 240$$
$$a\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = 240$$
$$\frac{a}{8} = 240$$
$$a = 1920$$

If $r = -\frac{1}{2}$, the first eight terms are 1920, -960, 480, -240, 120, -60, 30, and -15.

10. In a geometric sequence, the sum of the third and fourth terms is $\frac{3}{4}$ and the sum of the sixth and seventh terms is -6. Determine the first seven terms of this sequence.

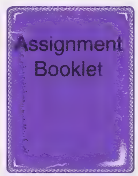
Check your answers by turning to the Appendix.

Conclusion

In this section you were shown how to identify sequences and investigate patterns in sequences. You also investigated arithmetic and geometric sequences. The terms of a sequence were determined from its defining rule. The general term formulas were derived for arithmetic and geometric sequences. You also applied the general term formulas to determine the various elements in a given sequence. Finally, you solved problems involving both arithmetic and geometric sequences.

Remember the football game you were introduced to at the beginning of this section? The quarterback calls a sequence of signals. Can you think of any other games where you have a sequence of signals? By now you should have more ideas about sequences in real-life situations.

Assignment



You are now ready to complete the section assignment.

Section 2: Series

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Millions of people across North America participate in baseball. They may participate as players (competitive or recreational), as coaches, as umpires, or as spectators.

Do you follow the World Series in Major League Baseball? For the last several years, games have been scheduled on alternate days, with longer breaks between games when the teams travel from one city to the other. When this sequence, of at most seven games, is viewed as a whole, it is referred to as a **series**. Similarly, in mathematics, the sum of the terms of a sequence is called a series.

In this section, you will investigate finite series. In particular, you will develop and apply formulas for both finite arithmetic and geometric series. Mortgages and annuities are but two examples of the practical application of series. In the last activity, you will be introduced to a shorthand device, called sigma notation, for representing series.

Let the series begin!

Activity 1: Identifying Series

If the commas in the general sequence $t_1, t_2, t_3, \ldots, t_n$ are replaced with plus signs, then $t_1 + t_2 + t_3 + \ldots + t_n$ is a series.

The domain of a sequence is N.

A **series** is the sum of the terms of a sequence. The **terms** of a series are also the terms of the sequence.

An **infinite series** is a series that has no last term. The general form of an infinite series is $t_1 + t_2 + t_3 + \ldots + t_n + \ldots$

A **finite series** has a last term. The general form of the finite series is $t_1 + t_2 + t_3 + \ldots + t_n$.

Is each series finite or infinite?

1. $6 + 10 + 14 + \ldots$
2. $3 + 5 + 7 + \ldots + 15$
3. $(-7) + (-2) + 3 + \ldots$
4. $5 + 10 + 15 + \ldots + 35$

Check your answers by turning to the Appendix.

Remember that t_n is the *n*th term (not the last term) in an infinite series; but t_n is the last term in the finite series.

Activity 2: Determining the Sum of Finite Series

The sum of the terms in a series can be determined by adding the terms if all the terms are given. To illustrate the sum of a series, the letter S with a subscipt is used to denote the number of terms that are to be added. The symbol S_4 means the sum of the first four terms. The sum of the first four terms is written as follows:

$$S_4 = t_1 + t_2 + t_3 + t_4$$

The sum of n terms is written as $S_n = t_1 + t_2 + t_3 + \ldots + t_n$.

Look at the next two examples.

Example 1

For the series $3 + 7 + 11 + \ldots$, determine the value of S_7.

Solution

$$\begin{aligned} S_7 &= 3 + 7 + 11 + 15 + 19 + 23 + 27 \\ &= 105 \end{aligned}$$

The expression for the first n number of terms of a series may be found by looking for a pattern.

Example 2

Find an expression for the sum of the first n terms of the series $1+3+5+7+\ldots$

Solution

Evaluate S_1, S_2, S_3, and S_4.

The first term, $S_1 = 1$ or 1^2.

The sum of the first two terms, $S_2 = 1+3 = 2^2$.

The sum of the first three terms, $S_3 = 1+3+5 = 3^2$.

The sum of the first four terms, $S_4 = 1+3+5+7 = 4^2$.

You will notice that the numbers obtained are perfect squares. This pattern will continue if more sums are evaluated. Therefore, the expression for the first n terms of the series is $S_n = n^2$.

Look at another example where the expression for the sum of the first n terms is known and the series can be found.

Example 3

If $S_n = n^2 - 2n$, find the first three terms of the series.

Solution

$$\begin{aligned} S_1 &= (1)^2 - 2(1) & \qquad t_1 &= -1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} S_2 &= (2)^2 - 2(2) & \qquad t_2 &= S_2 - S_1 \\ &= 0 & &= 0-(-1) \\ & & &= 1 \end{aligned}$$

$$\begin{aligned} S_3 &= (3)^2 - 2(3) & \qquad t_3 &= S_3 - S_2 \\ &= 3 & &= 3-0 \\ & & &= 3 \end{aligned}$$

Therefore, the first three terms are -1, 1, and 3.

1. Determine the first four terms of the series if $S_n = 2n^2 + n$.

Check your answers by turning to the Appendix.

The general term of the series can be found by subtracting the sum of $(n-1)$ terms from the sum of n terms if the expression for the sum of the first n terms of a series is known.

In any series, the general term is $t_n = S_n - S_{n-1}$, where $n > 1$.

Example 4

If $S_n = n^2 - 3n$, find the nth term.

Solution

$$\begin{aligned} t_n &= S_n - S_{n-1} \\ &= \left(n^2 - 3n\right) - \left[(n-1)^2 - 3(n-1)\right] \\ &= \left(n^2 - 3n\right) - \left[n^2 - 2n + 1 - 3n + 3\right] \\ &= \cancel{n^2} - \cancel{3n} - \cancel{n^2} + 2n - 1 + \cancel{3n} - 3 \\ &= 2n - 4 \end{aligned}$$

2. If $S_n = n^2 + n$, find the following:

   a. t_n b. t_6 c. S_6

Check your answers by turning to the Appendix.

In this activity, given the terms of a series, you found a formula for the sum of those terms. Conversely, given the formula for the sum, you wrote out the terms of the series.

Activity 3: Deriving the Sum Formula for Arithmetic Series

The numbers 7, 12, 17,…, 42 represent an arithmetic sequence.

The corresponding arithmetic series is $7 + 12 + 17 + \ldots + 42$.

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

1. State which of the following form an arithmetic series.

   a. $1 + 4 + 7 + \ldots + 28$
   b. $3 + 6 + 12 + 24 + 48$
   c. $\frac{1}{2} + 2 + 8 + 32 + 64$
   d. $3 + (-2) + (-7) + (-12) + (-17)$
   e. 6, 9, 12, 15, 18, 21, 24

Check your answers by turning to the Appendix.

Determining the sum of many terms in an arithmetic series can be a time-consuming process. A formula must be obtained to calculate the sum of n terms.

Consider the arithmetic series $3+5+7+9+11+13+15+17$. Now, write the terms in opposite order.

$17+15+13+11+9+7+5+3$

Add these series together.

$$\begin{aligned}
S_8 &= 3+5+7+9+11+13+15+17 \\
S_8 &= 17+15+13+11+9+7+5+3 \\
\hline
2S_8 &= 20+20+20+20+20+20+20+20 \\
2S_8 &= 8(20) \\
S_8 &= 80
\end{aligned}$$

This same method can be used to find the sum of any finite arithmetic series. Write the terms of the series twice. Add the columns to determine twice the sum. Then divide the double sum by two to obtain the proper sum.

Let S_n be the sum of the first n terms of an arithmetic series.

$$\begin{aligned}
S_n &= a+(a+d)+(a+2d)+\ldots+(t_n-d)+t_n \\
S_n &= t_n+(t_n-d)+(t_n-2d)+\ldots+(a+d)+a \\
\hline
2S_n &= (a+t_n)+(a+t_n)+(a+t_n)+\ldots+(a+t_n)+(a+t_n) && \text{(Add the columns.)} \\
2S_n &= n(a+t_n) && \text{(Combine the } (a+t_n) \text{ values.)} \\
S_n &= \frac{n}{2}(a+t_n) && \text{(Divide by 2.)}
\end{aligned}$$

In the second line, the order of the series is reversed.

The formula $\boldsymbol{S_n=\frac{n}{2}(a+t_n)}$ gives the sum of the first n terms of an arithmetic series if the first and last terms are known.

Another formula can be obtained from $S_n = \frac{n}{2}(a + t_n)$ by using the general expression for t_n.

Since $t_n = a + (n-1)d$, then substitute this into $S_n = \frac{n}{2}(a + t_n)$.

$$
\begin{aligned}
S_n &= \frac{n}{2}\{a + [a + (n-1)d]\} \\
&= \frac{n}{2}[a + a + (n-1)d] \\
&= \frac{n}{2}[2a + (n-1)d]
\end{aligned}
$$

Therefore, the form used if the last term is not known is $\mathbf{S_n = \frac{n}{2}[2a + (n-1)d]}$.

2. Determine the sum of the arithmetic series $5 + 9 + 13 + \ldots + 33$ by using the method at the beginning of this activity.

3. Check if the sum in question 1 is correct by using the sum formula $S_n = \frac{n}{2}(a + t_n)$.

4. Check if the sum is correct in question 1 by using the sum formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

Check your answers by turning to the Appendix.

Activity 4: Applying the Sum Formula of Arithmetic Series

The formulas for determining the sum of an arithmetic series were developed in Activity 4.

The sum formulas for an arithmetic series are as follows:

- $S_n = \frac{n}{2}(a + t_n)$
- $S_n = \frac{n}{2}[2a + (n-1)d]$

These formulas will now be used in the next four examples.

Example 1

Determine the sum of the first twelve terms of the arithmetic series $15 + 21 + 27 + \ldots$

Solution

$a = 15$, $d = 6$, and $n = 12$

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d] \\
&= \frac{12}{2}[2(15) + (12-1)6] \\
&= 6[30 + 11(6)] \\
&= 6[96] \\
&= 576
\end{aligned}
$$

The sum of the first twelve terms of the series is 576.

Example 2

Determine the sum of the following arithmetic series.

$$24+18+12+\ldots+(-78)$$

Solution

$a=24$, $d=-6$, and $t_n=-78$

The number of terms n must be determined before the sum formulas can be used.

$$\begin{aligned} t_n &= a+(n-1)d \\ -78 &= 24+(n-1)(-6) \\ -78 &= 24-6n+6 \\ -78 &= 30-6n \\ 6n &= 108 \\ n &= 18 \end{aligned}$$

Now use the sum formula.

$$\begin{aligned} S_n &= \frac{n}{2}(a+t_n) \\ S_{18} &= \frac{18}{2}[24+(-78)] \\ &= 9[-54] \\ &= -486 \end{aligned}$$

The sum of the series is -486.

1. Determine the sum of each arithmetic series given the first term and the last term. The position number is also given for the last term as shown.

   a. $a=-9$ and $t_{14}=56$

   b. $a=12$ and $t_{16}=-108$

   c. $a=21$ and $t_{11}=-9$

   d. $a=-40$ and $t_{32}=-195$

2. For each arithmetic series, calculate the required sum.

   a. $1+4+7+\ldots$
   Determine S_{15} (the sum of the first fifteen terms).

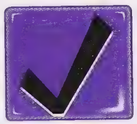
   b. $(-22)+(-18)+(-14)+\ldots$
   Determine S_9.

   c. $19+16+13+\ldots$
   Determine S_{18}.

   d. $(-14)+(-17)+(-20)+\ldots$
   Determine S_{12}.

3. Determine the sum of all the multiples of 8 between 20 and 205.

4. Determine the sum of the arithmetic series $\frac{1}{3}+\frac{7}{3}+\frac{13}{3}+\ldots+\frac{43}{3}$.



Check your answers by turning to the Appendix.

Example 3

The sum of the first ten terms of an arithmetic series is 20, and the sum of the first eighteen terms of this same series is 324. Determine the common difference, the first term, and the sum of the first fourteen terms.

Solution

$S_{10} = 20$ and $n = 10$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$20 = \frac{10}{2}[2a + (10-1)d]$$

$$20 = 5[2a + 9d] \quad \text{(Divide both sides by 5.)}$$

$$4 = 2a + 9d \qquad (1)$$

$S_{18} = 324$ and $n = 18$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$324 = \frac{18}{2}[2a + (18-1)d]$$

$$324 = 9[2a + 17d] \quad \text{(Divide both sides by 9.)}$$

$$36 = 2a + 17d \qquad (2)$$

Subtract (1) from (2).

$$\begin{aligned} 36 &= 2a + 17d \\ \underline{4} &\underline{= 2a + 9d} \\ 32 &= 8d \\ d &= 4 \end{aligned}$$

The common difference is 4.

Substitute $d = 4$ in (1).

$$\begin{aligned} 4 &= 2a + 9(4) \\ -32 &= 2a \\ a &= -16 \end{aligned}$$

The first term is -16.

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ S_{14} &= \frac{14}{2}[2(-16) + (14-1)(4)] \\ &= 7[-32 + 13(4)] \\ &= 7[20] \\ &= 140 \end{aligned}$$

The sum of the first fourteen terms is 140.

5. The sum of the first fourteen terms of an arithmetic series is -28, and the sum of the first twenty-six terms of this same series is 572. Determine the common difference, the first term, and the sum of the first fifty terms.

Check your answers by turning to the Appendix.

Example 4

Determine the number of terms in the arithmetic series $16+11+6+\ldots$ which have a sum of -36.

Solution

$a=16$, $d=-5$, and $S_n=-36$

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

$$-36=\frac{n}{2}[2(16)+(n-1)(-5)]$$

$$-36=\frac{n}{2}[32-5n+5]$$

$$-36=\frac{n}{2}[37-5n]$$

$$-72=n(37-5n)$$

$$-72=37n-5n^2$$

$$5n^2-37n-72=0$$

$$(5n+8)(n-9)=0$$

$$5n+8=0 \quad\text{or}\quad n-9=0$$

$$5n=-8 \qquad n=9$$

$$n=-\frac{8}{5}$$

Since n must be a natural number, $n=-\frac{8}{5}$ is inadmissible.

Thus, 9 terms in the arithmetic series have a sum equal to -36.

6. Determine the number of terms in each arithmetic series. The sum of the terms is given to the right of the series.

   a. $1+2+3+\ldots \quad (S_n=210)$

   b. $(-20)+(-15)+(-10)+\ldots \quad (S_n=90)$

   c. $11+8+5+\ldots \quad (S_n=-184)$

7. How many consecutive, odd natural numbers should be added to obtain the sum of 289? The first term of this arithmetic series is 1.

Check your answers by turning to the Appendix.

The arithmetic series is used in many problem-solving situations. Study Example 5; then try the questions.

Example 5

A chemistry lecture room has 17 seats in the first row and 2 seats more in each following row. Determine the number of seats in the front twelve rows.

Solution

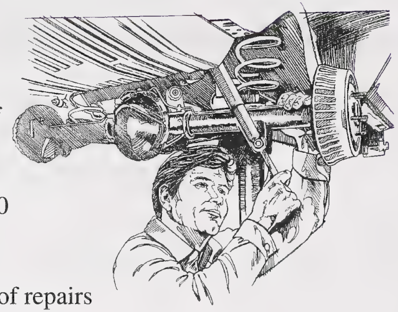
The number of seats in each row form the arithmetic series $17+19+21+\dots$

$a=17,\ d=2,$ and $n=12$

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a+(n-1)d] \\
S_{12} &= \frac{12}{2}[2(17)+(12-1)2] \\
&= 6[34+11(2)] \\
&= 6[34+22] \\
&= 6[56] \\
&= 336
\end{aligned}
$$

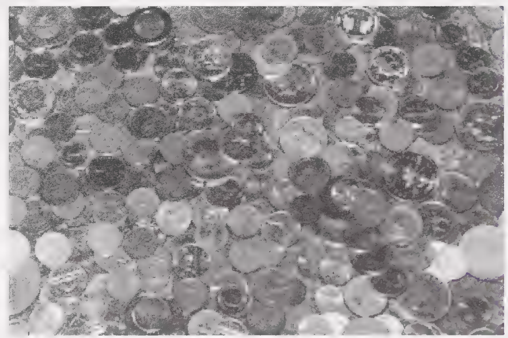
There are 336 seats in the first twelve rows of the chemistry lecture room.

8. At the end of the first year, the cost of repairing a truck is \$160. The yearly cost of repairs increases \$80 each year. For example, the cost of repairs is \$240 for the second year and \$320 for the third year. Determine the total cost of repairs if this vehicle is kept for eight years.



9. Fence posts are stacked such that there are 26 posts in the bottom row, 25 in the next row, 24 in the next row, and so on. There are 10 posts in the top row. Determine the total number of posts.

10. A child saves \$0.35 the first week, \$0.40 the second week, \$0.45 the third, and so on. For how many weeks did the child save if this child saved a total of \$32.25?



11. A motorcycle costs $5500. The motorcycle depreciates 12% the first year. The depreciation for each following year is $70 more than for the previous year. The value of the motorcycle at the end of five years is the initial cost minus the sum of the depreciations for the five years. What is the value of the motorcycle after five years?

12. An object is dropped from a hovering helicopter. The object falls 4.9 m in the first second, 14.7 m in the next second, 24.5 m in the third second, and so on.

    a. What distance does the object fall in the tenth second, just before hitting the ground?

    b. What is the total distance the object falls? (Recall that the object hit the ground in 10 s.) What is the height of the helicopter above the ground?

Check your answers by turning to the Appendix.

You solved problems involving the use and application of the sum formula for arithmetic series. Did you find the real-life problems interesting?

Activity 5: Deriving the Sum Formula for Geometric Series

The numbers $5, \ -15, \ 45, \ldots, \ -1215$ represent a geometric sequence.

The corresponding geometric series is $5+(-15)+45+\ldots+(-1215)$.

A **geometric series** is the sum of the terms of a geometric sequence.

1. State which of the following form a geometric series.

    a. $-3+(-6)+(-9)+\ldots+(-24)$

    b. $-2+4+(-8)+16+(-32)$

    c. $\frac{1}{8}+\left(-\frac{1}{4}\right)+\frac{1}{2}+(-1)+2+(-4)$

    d. $-14+(-8)+(-2)+4+10+16$

    e. $-12, \ 6, \ -3, \ \frac{3}{2}, \ -\frac{3}{4}$

Check your answers by turning to the Appendix.

In order to find the formula for the sum of a geometric series, consider the geometric series $5+10+20+\ldots+160$. For this series, $a=5$ and $r=2$. Let S be the sum of this series.

$$S = 5 + 10 + 20 + \ldots + 160$$
$$2S = \quad 10 + 20 + \ldots + 160 + 320$$

Rewrite these two expressions so that the $2S$ expression is above the S expression, and put the same terms in the same vertical columns.

Multiply the series by the common ratio r, which is 2.

$$\begin{array}{rl} 2S = & \quad 10 + 20 + \ldots + 160 + 320 \\ S = & 5 + 10 + 20 + \ldots + 160 \\ \hline 2S - S = & -5 \qquad\qquad\qquad\quad +320 \end{array}$$

(Subtract S from $2S$.)

$$S = -5 + 320$$
$$= 315$$

The sum of the geometric series is 315.

A similar method can be used to find the sum of a general geometric series with a common ratio r. Let the sum of n terms be S_n.

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$$

S_n is now multiplied by the common ratio r giving rS_n.

$$rS_n = \quad ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n$$

The position of the S_n and rS_n are changed so that the rS_n expression is above the S_n expression.

$$\begin{array}{rl} rS_n = & \quad ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \\ S_n = & a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \\ \hline rS_n - S_n = & -a + 0 + 0 + \ldots + 0 + 0 + ar^n \end{array}$$

$$S_n(r-1) = ar^n - a$$
$$S_n(r-1) = a(r^n - 1)$$

Divide both sides by $r-1$.

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}}$$

This formula can be used to find the sum of n terms of a geometric series for any value of r except $r = 1$. This formula is usually used when $|r| > 1$. A different version of this formula is used when $|r| < 1$ in order to simplify calculations.

Both formulas for finding the sum (S_n) of a geometric series containing n terms are as follows:

$$S_n = \frac{a(r^n - 1)}{r - 1}, \text{ when } |r| > 1$$

$$S_n = \frac{a(1 - r^n)}{1 - r}, \text{ when } |r| < 1$$

Consider the series $20 + 20 + 20 + 20 + 20$. If the type of series is not stated, then the series could be arithmetic where $a = 20$ and $d = 0$. The series could also be geometric where $a = 20$ and $r = 1$. If the series is considered arithmetic, then the sum of the series can be determined by using the formula $S_n = \frac{n}{2}(a + t_n)$.

$$S_n = \frac{n}{2}(a + t_n)$$
$$S_5 = \frac{5}{2}(20 + 20)$$
$$= 100$$

If the series is considered geometric, then the formulas given previously cannot be used since division by zero is undefined. When $r = 1$ for a geometric series, then the sum can be represented by the formula $S_n = na$.

$$S_n = na$$
$$S_5 = 5(20)$$
$$= 100$$

2. Determine the sum of the geometric series $(-3)+(-12)+(-48)+\ldots+(-3072)$ by using the method at the beginning of this activity.

3. Check if the sum in question 1 is correct by using the appropriate sum formula for the geometric series.

Check your answers by turning to the Appendix.

Since you are familiar with the sum formula for geometric series, you will solve some problems in the following activity.

Activity 6: Applying the Sum Formula of Geometric Series

You have just discovered that the sum formulas for a geometric series are $S_n = \frac{a(r^n - 1)}{r-1}$, when $|r| > 1$, and $S_n = \frac{a(1-r^n)}{1-r}$, when $|r| < 1$.

Study the next examples using these sum formulas.

Example 1

Determine the sum of the first ten terms of the geometric series $6 + 12 + 24 + \ldots$

Solution

$a = 6$, $n = 10$, and $r = 2$

Since $|r| > 1$, $S_n = \frac{a(r^n - 1)}{r - 1}$

$$\begin{aligned} S_{10} &= \frac{6(2^{10} - 1)}{2 - 1} \\ &= 6(1024 - 1) \\ &= 6(1023) \\ &= 6138 \end{aligned}$$

1. **a.** Determine S_9 of $18+6+2+\ldots$

   b. Determine S_8 of $(-4)+8+(-16)+\ldots$

   c. Determine S_{10} of $(-2)+(-6)+(-18)+\ldots$

   d. Determine S_5 of $100+(-20)+4+\ldots$

Check your answers by turning to the Appendix.

Example 2

Determine the sum of the geometric series $45+15+5+\ldots+\frac{5}{27}$.

Solution

$$t_n = \frac{5}{27},\ a = 45,\ \text{and } r = \frac{5}{15}$$
$$= \frac{1}{3}$$

The value of n must be determined first.

$$\begin{aligned}
t_n &= ar^{n-1} \\
\frac{5}{27} &= 45\left(\frac{1}{3}\right)^{n-1} \\
\left(\frac{5}{27}\right)\left(\frac{1}{45}\right) &= \left(\frac{1}{3}\right)^{n-1} \\
\frac{1}{243} &= \left(\frac{1}{3}\right)^{n-1} \\
\frac{1}{3^5} &= \frac{1}{3^{n-1}} \\
5 &= n-1 \\
6 &= n
\end{aligned}$$

Since $|r|<1$, $S_n = \dfrac{a(1-r^n)}{1-r}$

$$\begin{aligned}
S_6 &= \frac{45\left[1-\left(\frac{1}{3}\right)^6\right]}{1-\frac{1}{3}} \\
&= \frac{45\left[1-\frac{1}{729}\right]}{\frac{2}{3}} \\
&= \frac{3}{2}(45)\left[\frac{728}{729}\right] \\
&= \frac{1820}{27} \text{ or } 67\frac{11}{27}
\end{aligned}$$

2. Determine the sums of these geometric series.

a. $(-8)+24+(-72)+\ldots+1944$

b. $36+(-18)+9+\ldots+\left(-\frac{9}{8}\right)$

Check your answers by turning to the Appendix.

Example 3

The sum of a certain number of terms in the geometric series $3+(-6)+12+\ldots$ is -255. How many terms produce this sum?

Solution

$$S_n = -255,\ a=3,\ \text{and } r = \frac{-6}{3} = -2$$

$$\begin{aligned}
\text{Since } |r|>1,\ S_n &= \frac{a\left(r^n-1\right)}{r-1} \\
-255 &= \frac{3\left[(-2)^n-1\right]}{-2-1} \\
-255 &= \frac{3\left[(-2)^n-1\right]}{-3} \\
255 &= (-2)^n-1 \\
256 &= (-2)^n \\
(-2)^8 &= (-2)^n \\
n &= 8
\end{aligned}$$

The first 8 terms in this geometric series will have a sum of -255.

3. For the geometric series $4+(-12)+(36)+\ldots$, how many terms produce a sum of 2188?

Check your answers by turning to the Appendix.

Example 4

If the sum of the first seven terms is 21.5 and the common ratio is -2 in a geometric series, determine the first term.

Solution

$S_7 = 21.5$, $n = 7$, and $r = -2$

$$\text{Since } |r| > 1,\ S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

$$21.5 = \frac{a\left[(-2)^7 - 1\right]}{-2-1}$$

$$21.5 = \frac{a[-128-1]}{-3}$$

$$-64.5 = a[-129]$$

$$a = \frac{-64.5}{-129}$$

$$= \frac{1}{2}$$

The first term is $\frac{1}{2}$.

4. Determine the first term in a geometric series if the sum of the first five terms is $-11\frac{5}{8}$ and the common ratio is $\frac{1}{2}$.

5. A geometric series has $t_7 = 243$ and $r = 3$. Determine the sum of the first eight terms in this series.

Check your answers by turning to the Appendix.

The following example illustrates a partial sum of an infinite geometric series. This means the required sum represents a specific number of terms within an infinite series.

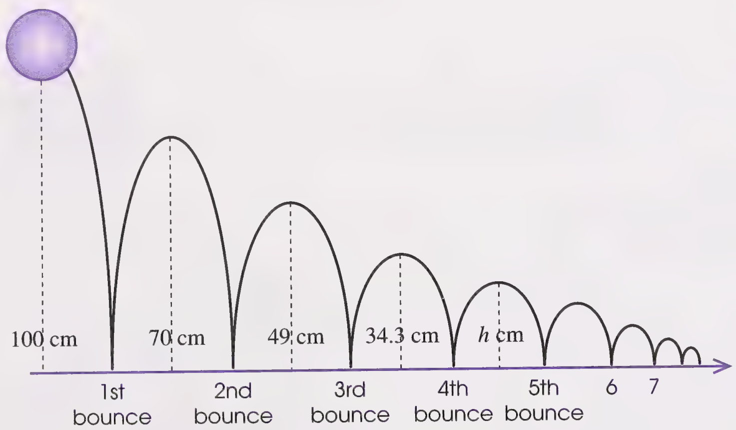
Example 5

A hard rubber ball is dropped from a height of 100 cm above a concrete floor. On each bounce the ball reaches a vertical height that is 0.7 times the previous vertical height.

- Determine the vertical height of the ball after the fourth bounce.
- Determine the total vertical distance travelled by the ball when it contacts the floor for the fifth time.

Solution

In the diagram the broken lines represent the vertical heights.



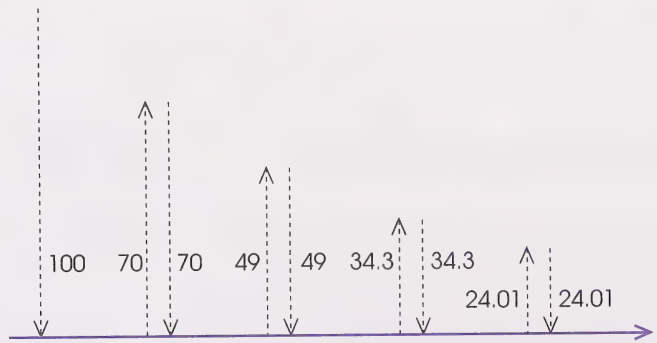
The vertical heights represent the terms of a geometric sequence where $a = 100$ and $r = 0.7$. The height of the ball after the fourth bounce is represented by h on the diagram. This height h is the 5th term of the sequence.

Thus, use $t_n = ar^{n-1}$.

$$t_5 = 100(0.7)^4$$
$$h = 100(0.2401)$$
$$= 24.01$$

The vertical height of the ball after the fourth bounce is 24.01 cm.

When calculating the total vertical distance travelled, note that the ball will travel upwards then downwards the same distance after each bounce. The total distance travelled will be the sum of the upward distances and the downward distances.



Note that there is only one vertical height of 100 cm when the ball is dropped from that height.

The total vertical distance travelled by the ball will be $100 + 2S_n$, where S_n represents the sum of the required terms starting with 70. Thus, $a = 70$, $n = 4$, and $r = 0.7$.

Since $|r|<1$, $S_n = \dfrac{a(1-r^n)}{1-r}$

$$\begin{aligned} S_4 &= \frac{70\left[1-(0.7)^4\right]}{1-0.7} \\ &= \frac{70[1-0.2401]}{0.3} \\ &= \frac{70[0.7599]}{0.3} \\ &= 177.31 \end{aligned}$$

$$\begin{aligned} \text{total vertical distance travelled} &= 100 + 2S_n \\ &= 100 + 2(177.31) \\ &= 100 + 354.62 \\ &= 454.62 \end{aligned}$$

The total vertical distance travelled is 454.62 cm.

6. The first prize in a lottery is $400 000. Second prize is $\frac{2}{5}$ of the first prize, third prize is $\frac{2}{5}$ of the second prize, and so on.

   a. Determine the value of the fifth prize.

   b. Determine the total amount of money for the first seven prizes.

7. A pensioner receives payments from a pension every six months. If the first payment is $4000 and the second payment is 0.95 times the previous amount. What is the total amount that this pensioner will have received at the end of three years? The geometric series is $4000 + 3800 + 3610 + \ldots$, where $3800 is the amount received at the end of the first year.

8. A runner is jogging along an incline and runs 10 km in the first hour. For each of the next three hours, the jogger attains 0.6 times the distance run during the previous hour. Determine the total distance covered by the jogger at the end of 4 h.

Check your answers by turning to the Appendix.

You should now be able to solve problems involving the use and application of the sum formula for geometric series.

Activity 7: Using Sigma Notation

The Greek letter $\sum$, called **sigma**, is a summation symbol.

The arithmetic series $5+8+11+14+17+20+23$ has $a=5$ and $d=3$.

The general term formula is $t_n = a+(n-1)d$.

When $a=5$, and $d=3$, $t_n = 5+(n-1)3$.

$$t_n = 5+3n-3$$
$$t_n = 2+3n$$

The general term is $2+3n$, and the abbreviation is $\sum_{n=1}^{7} 2+3n$.

This is read as **the summation from 1 to 7 of $2+3n$.**

The summation notation means that the seven terms in this series are to be added. The terms are obtained from the general term $2+3n$ by replacing n with the consecutive integers from 1 to 7 inclusive. The number at the bottom of $\sum$ shows the value of n in the first term of the series and the number at the top of $\sum$ shows the value of n in the last term.

In this section, the value of n is always a natural number (positive integer). If n is an exponent in the general term, then the series is usually a geometric series. If n is not an exponent in the general term, then the series is usually an arithmetic series.

Now study the following examples.

Example 1

Expand the following series.

$$\sum_{n=3}^{7} 5n-8$$

Solution

Substitute 3, 4, 5, 6, and 7 for n in $5n-8$.

$$\begin{aligned}\sum_{n=3}^{7} 5n-8 &= [5(3)-8]+[5(4)-8]+[5(5)-8]+[5(6)-8] \\ &\quad +[5(7)-8] \\ &= [15-8]+[20-8]+[25-8]+[30-8]+[35-8] \\ &= 7+12+17+22+27\end{aligned}$$

$$\sum_{n=3 \leftarrow \text{ first substitution}}^{7 \leftarrow \text{ last substitution}}$$

Example 2

Expand the following series.

$$\sum_{k=3}^{5} 2^{k+1}$$

Solution

Substitute 3, 4, and 5 for 2^{k+1}.

$$\begin{aligned}\sum_{k=3}^{5} 2^{k+1} &= \left[2^{3+1}\right]+\left[2^{4+1}\right]+\left[2^{5+1}\right] \\ &= 2^4+2^5+2^6\end{aligned}$$

1. Expand each of the following series.

a. $\sum_{n=2}^{6} 4-n$ **b.** $\sum_{k=2}^{5} 3^{k^2-1}$

c. $\sum_{n=4}^{7} (n-2)^2$ **d.** $\sum_{n=3}^{6} 2^n$

Check your answers by turning to the Appendix.

Example 3

Find the sum for $\sum_{n=1}^{5} n^2-1$.

Solution

Substitute $n=1$, 2, 3, 4, and 5 in $\left(n^2-1\right)$ to determine the five terms of the series.

$$\begin{aligned}S_5 &= \left(1^2-1\right)+\left(2^2-1\right)+\left(3^2-1\right)+\left(4^2-1\right)+\left(5^2-1\right) \\ &= 0+3+8+15+24 \\ &= 50\end{aligned}$$

2. Determine the sum for each of the following series. You may use a calculator to obtain the sums.

a. $\sum_{n=1}^{6} 2n-3$ **b.** $\sum_{k=3}^{7} 5k+2$ **c.** $\sum_{n=1}^{5} 3^n$

d. $\sum_{k=2}^{6} (k-4)^2$ **e.** $\sum_{n=1}^{4} -4^n+(-4)^n$

Check your answers by turning to the Appendix.

Example 4

Write the series $4+7+10+\ldots+19$ in sigma notation.

Solution

This is an arithmetic series where the common difference is 3.

$$\begin{aligned} t_n &= a+(n-1)d \\ &= 4+(n-1)(3) \\ &= 4+3n-3 \\ &= 3n+1 \end{aligned}$$

The number of the last term must be determined.

$$\begin{aligned} t_n &= 3n+1 \\ 19 &= 3n+1 \\ 18 &= 3n \\ n &= 6 \end{aligned}$$

The summation notation is $\sum_{n=1}^{6} 3n+1$.

Example 5

Write the series $2+6+18+\ldots+1458$ in summation notation.

Solution

This is a geometric series where $r=\frac{18}{6}=3$.

$$t_n = ar^{n-1}$$

Substitute $a=2$ and $r=3$.

$$t_n = 2(3)^{n-1}$$

The number of terms must be determined.

$$\begin{aligned} 1458 &= 2(3)^{n-1} \\ 729 &= 3^{n-1} \\ 3^6 &= 3^{n-1} \\ 6 &= n-1 \\ n &= 7 \end{aligned}$$

The summation notation is $\sum_{n=1}^{7} 2(3)^{n-1}$.

3. Write each series in sigma notation.

a. $3+5+7+9+\ldots+25$

b. $3+15+75+\ldots+1875$

c. $\frac{1}{4}+\frac{1}{2}+1+\ldots+128$

d. $-4+(-2)+0+2+\ldots+18$

e. $3+12+27+48+75+108+147$

Check your answers by turning to the Appendix.

To write a series in summation notation, you had to identify it as an arithmetic or geometric series. Then you can find the number of terms by using the correct general term formula.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The symbol $\sum_{k=1}^{n} a_k$ has two meanings. It may denote either the series $a_1+a_2+a_3+\ldots+a_n$ or the sum of the series. The directions will always be clear as to which meaning is intended.

1. Expand the following series.

a. $\sum_{n=6}^{10} 4n+5$

b. $\sum_{k=3}^{7} \frac{2}{k+1}$

2. Determine the sum for each series.

a. $\sum_{n=3}^{5} 2^{n+1}$

b. $\sum_{k=7}^{9} k^2-6k$

Check your answers by turning to the Appendix.

You may find it helpful to review the concept of series by viewing the video titled *Arithmetic Sequences and Series* from the *Catch 30* series, ACCESS Network. This video is available from the Learning Resources Distributing Centre.

If you take the arithmetic series $2+4+6+\ldots+16$, you can find the sum of any number of terms.

Let S_2 be the sum of the first two terms, S_3 be the sum of the first three terms, and so on.

$$\begin{aligned} \therefore\ S_2 &= 2+4=6 \\ S_3 &= 2+4+6=12 \\ S_5 &= 2+4+6+8+10=30 \end{aligned}$$

This process can become quite lengthy. This necessitated the development of the sum formula $S_n = \dfrac{n(a+t_n)}{2}$.

3. Form the sequence S_1, S_2, S_3, …, S_6 for the series $16+9+2+\ldots$

4. Determine the number of terms in the arithmetic series $(-28)+(-23)+(-18)+\ldots$ which have a sum of -55.

Check your answers by turning to the Appendix.

In Activity 5 of this section you used the two sum formulas for the geometric series, depending on the common ratio ($r>1$ or $r<1$).

However, you can use the formula $S_n = \frac{a(r^n-1)}{r-1}$ for both cases, $r>1$ or $r<1$.

You may find it helpful to review the concept of series by viewing the video titled *Geometric Sequences and Series* from the *Catch 30* series, ACCESS Network. This video is available from the Learning Resources Distributing Centre.

Example

Determine the sum of the first eight terms of the geometric series $16+4+1+\ldots$ to the nearest tenth.

Solution

$a=16$, $n=8$, and $r=\dfrac{4}{16}=\dfrac{1}{4}$

$$\begin{aligned} S_n &= \frac{a(r^n-1)}{r-1} \\ &= \frac{16\left[\left(\frac{1}{4}\right)^8-1\right]}{\frac{1}{4}-1} \\ &= \frac{16\left[\frac{1}{65\,536}-1\right]}{-\frac{3}{4}} \\ &= \left(-\frac{4}{3}\right)(16)\left(-\frac{65\,535}{65\,536}\right) \\ &= 21.3 \text{ (to the nearest tenth)} \end{aligned}$$

5. Find the sum of six terms of the geometric series $2+1+\frac{1}{2}+\dots$

6. How many terms produce the geometric series $243+81+27+\dots$ if the sum is 360?

7. Determine the sum of the first seven terms of the geometric series where $t_7=\frac{1}{8}$ and $r=-\frac{1}{2}$.

Check your answers by turning to the Appendix.

Enrichment

There are a number of properties of series that are best expressed using sigma notation. For instance, the following example illustrates that a constant factor may be removed to simplify the sum before it is evaluated. This example is a specific instance of the more general theorem, $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$.

Example

Prove $\sum_{k=1}^{5} ck = c\sum_{k=1}^{5} k$.

Solution

LS	RS
$\sum_{k=1}^{5} ck$	$c\sum_{k=1}^{5} k$
$=c+2c+3c+4c+5c$	$=c(1+2+3+4+5)$
$=15c$	$=c(15)$
	$=15c$

LS = RS

Thus, $\sum_{k=1}^{5} ck = c\sum_{k=1}^{5} k$.

1. Prove that $\sum_{n=1}^{4}[3n+2n]=\sum_{n=1}^{4}3n+\sum_{n=1}^{4}2n$.

2. Find the sum of the series $\sum_{k=1}^{3}(-1)^{k+1}(2k)$.

3. Colin is a soccer player. In order to condition himself, Colin runs each day. On the first day, he runs 5 km. He increases by 2 km each day until he runs 25 km per day. Write the sum of the total distance Colin has run in summation notation. Then find the sum.

Check your answers by turning to the Appendix.

You may use the spreadsheet application, Microsoft Works™ or Claris Works® to determine the sum of a geometric series.

Set up the spreadsheet as shown.

	A	B	C	D
1	Sum of Geometric Series = a*(r^n-1)/(r-1)			
2				
3	S(sum)	a	r	n
4				

Enter the following numbers.

$B4 = 4$

$C4 = 2$

$D4 = 3$

In Cell A 4, enter the following formula.

= B4 * (C4 ^ D4 - 1)/(C4 - 1)

	A	B	C	D
1	Sum of Geometric Series = a*(r^n-1)/(r-1)			
2				
3	S(sum)	a	r	n
4	28	4	2	3

The formula will not be displayed in Cell A 4, but the sum 28 will be displayed.

Copy the formula from A 4 into cells A5 to A9.

	A	B	C	D
1	Sum of Geometric Series = a*(r^n-1)/(r-1)			
2				
3	S(sum)	a	r	n
4	28	4	2	3
5		6	3	2
6		5	4	3
7		10	3	5
8		25	4	2
9				

Did you get the following values?

$A5 = 24$

$A6 = 105$

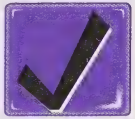
$A7 = 1210$

$A8 = 125$

You may choose your own values for a, r, and n to get more practice.

4. Use the spreadsheet to find the value of the first term for a geometric series when $S_n = 560$, $r = 3$, and $n = 4$.

5. Determine the first term of a geometric series when $S_n = -2520$, $r = 2$, and $n = 6$.



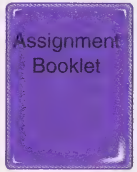
Check your answers by turning to the Appendix.

Conclusion

In this section you defined series as the sum of the terms of a sequence. You developed and applied formulas for both arithmetic and geometric series. Because series are based on sequences, you had to use what you had studied about sequences in Section 1. In Activity 7, you applied a short-hand device, called sigma notation, for representing series. In each activity you applied series to a variety of real-world problems.

In mathematics, the meaning of series is precise, and is different from, although based upon, the concept of sequence. In everyday conversation, however, the words *sequence* and *series* are often used interchangeably. Nevertheless, when the word *series* is used, the idea of a group or whole is implied. The World Series, for example, refers to the four to seven games played at the end of the regular season for baseball supremacy in North America. The World Series means much more to baseball fans than simply the way the individual games are scheduled! To some, it represents a way of life.

Assignment



You are now ready to complete the section assignment.

Module Summary

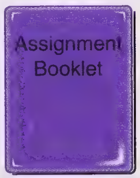
In this module, you explored both sequences and series.

In Section 1, you investigated arithmetic and geometric sequences—developing and applying the general-term formulas, not only in a purely mathematical context, but in a variety of real-world settings (such as the calculation of compound interest).

In Section 2, you explored arithmetic and geometric series—developing and applying the formulas for the sums of their terms. Again, you looked at a number of practical applications. For instance, the formulas for annuities and mortgages are based on geometric series. Finally, you looked at sigma notation as a notational device for representing series.

In a fundamental way, the topic of sequences and series represents what mathematics is all about—the search for patterns. To survive in a large city, like the one in the photograph, to find your way around, you need to recall and locate addresses, and how those addresses are laid out. Most people rely on number patterns to help them remember. Have you experienced the frustration of trying to find your way around a city that uses street names rather than numbers?

Assignment



You are now ready to complete the module assignment.

APPENDIX

Glossary

Suggested Answers

Glossary

arithmetic means: the terms between any two terms of an arithmetic sequence

arithmetic sequence: a sequence where each term is formed from the preceding term by adding a constant to the previous term

arithmetic series: the sum of the terms of an arithmetic sequence

common difference: the constant which is added to each term to produce the arithmetic sequence

common ratio: a constant which is used to multiply each term to produce the geometric sequence

finite sequence: a sequence that has a last term

finite series: a series that has a last term

geometric means: the terms between any two terms of a geometric sequence

geometric sequence: a sequence where each term is obtained by multiplying the preceding term by a constant

geometric series: the sum of the terms of a geometric sequence

infinite sequence: a sequence that has no last term

infinite series: a series that has no last term

recursion formula: a formula used to define each term with reference to the preceding term

sequence: a set of numbers arranged in a definite order

series: the sum of the terms of a sequence

sigma: $\left(\sum\right)$ a Greek letter used as a summation symbol

Suggested Answers

Section 1: Activity 1

1. **a.** infinite **b.** finite
   c. finite **d.** infinite
   e. finite **f.** finite
   g. finite **h.** infinite
   i. infinite **j.** infinite

2. **a.** When $n=1$,

$$\begin{aligned} t_1 &= -3(1)^2 + 2(1) \\ &= -3(1) + 2 \\ &= -1 \end{aligned}$$

When $n=2$,

$$\begin{aligned} t_2 &= -3(2)^2 + 2(2) \\ &= -3(4) + 4 \\ &= -8 \end{aligned}$$

When $n=3$,

$$t_3=-3(3)^2+2(3) \\ =-3(9)+6 \\ =-21$$

When $n=4$,

$$t_4=-3(4)^2+2(4) \\ =-3(16)+8 \\ =-40$$

The first four terms are -1, -8, -21, and -40.

b. When $n=1$,

$$t_1=4(1)^2-10(1) \\ =4(1)-10 \\ =-6$$

When $n=2$,

$$t_2=4(2)^2-10(2) \\ =4(4)-20 \\ =-4$$

When $n=3$,

$$t_3=4(3)^2-10(3) \\ =4(9)-30 \\ =6$$

When $n=4$,

$$t_4=4(4)^2-10(4) \\ =4(16)-40 \\ =24$$

The first four terms are -6, -4, 6, and 24.

3. a. When $h=1$,
$f:1\rightarrow -4(1)+7=3$

When $h=2$,
$f:2\rightarrow -4(2)+7=-1$

When $h=3$,
$f:3\rightarrow -4(3)+7=-5$

When $h=4$,
$f:4\rightarrow -4(4)+7=-9$

The first four terms are 3, -1, -5, and -9.

b. When $h=1$,
$f:1\rightarrow 5(1)-2=3$

When $h=2$,
$f:2\rightarrow 5(2)-2=8$

When $h=3$,
$f:3\rightarrow 5(3)-2=13$

When $h=4$,
$f:4\rightarrow 5(4)-2=18$

The first four terms are 3, 8, 13, and 18.

4. When $n=1$,

$$t_1=-4(1)^2+15 \\ =-4(1)+15 \\ =11$$

When $n=2$,

$$t_2=-4(2)^2+15 \\ =-4(4)+15 \\ =-1$$

When $n=3$,

$$t_3=-4(3)^2+15 \\ =-4(9)+15 \\ =-21$$

When $n=4$,

$$t_4=-4(4)^2+15 \\ =-4(16)+15 \\ =-49$$

The sequence is $11,\ -1,\ -21,\ -49,\ldots,\left(-4n^2+15\right),\ldots$

5. When $n=1$,

$$t_1=8(1)-2 \\ =6$$

When $n=2$,

$$t_2=8(2)-2 \\ =14$$

When $n=3$,

$$\begin{aligned} t_3 &= 8(3)-2 \\ &= 22 \end{aligned}$$

When $n=4$,

$$\begin{aligned} t_4 &= 8(4)-2 \\ &= 30 \end{aligned}$$

When $n=12$,

$$\begin{aligned} t_{12} &= 8(12)-2 \\ &= 94 \end{aligned}$$

The sequence is 6, 14, 22, 30,…, $(8n-2)$,…, 94.

Section 1: Activity 2

1. Each successive term is obtained by subtracting **4** from the preceding term; or each successive term is obtained by adding **−4** to the preceding term.

$$\begin{aligned} t_5 &= -2+(-4) \\ &= -6 \end{aligned} \qquad \begin{aligned} t_6 &= -6+(-4) \\ &= -10 \end{aligned}$$

2. Each successive term is obtained by **multiplying** the previous term by **2**.

$$\begin{aligned} t_5 &= 2(8) \\ &= 16 \end{aligned} \qquad \begin{aligned} t_6 &= 2(16) \\ &= 32 \end{aligned}$$

3. Each successive term is obtained by multiplying the preceding term by $\mathbf{\frac{1}{4}}$; or each successive term is obtained by dividing the preceding term by **4**.

$$\begin{aligned} t_4 &= \frac{1}{4}\left(\frac{1}{16}\right) \\ &= \frac{1}{64} \end{aligned} \qquad \begin{aligned} t_5 &= \frac{1}{4}\left(\frac{1}{64}\right) \\ &= \frac{1}{256} \end{aligned}$$

4. Each successive term is obtained by multiplying the preceding term by **2**.

$$\begin{aligned} t_4 &= 2\left(4\sqrt{5}\right) \\ &= 8\sqrt{5} \end{aligned} \qquad \begin{aligned} t_5 &= 2\left(8\sqrt{5}\right) \\ &= 16\sqrt{5} \end{aligned}$$

5. Multiply the previous term by $\sqrt{\mathbf{3}}$.

$$\begin{aligned} t_4 &= 9\left(\sqrt{3}\right) \\ &= 9\sqrt{3} \end{aligned} \qquad \begin{aligned} t_5 &= \sqrt{3}\left(9\sqrt{3}\right) \\ &= 27 \end{aligned}$$

6. a.

n	1	2	3	4
t_n	4	6	8	10

From the table of values, an increase of 1 in n produces an increases of 2 for t_n. Thus, $t_n = 2n + b$.

Use the ordered pair $(1, 4)$ to find b.

$$4 = 2(1) + b$$
$$2 = b$$

Therefore, $t_n = 2n + 2$.

$$t_5 = 2(5) + 2 \qquad t_6 = 2(6) + 2$$
$$= 12 \qquad\qquad = 14$$

b.

n	1	2	3	4
t_n	−5	−8	−11	−14

From the table of values, an increase of 1 in n produces a decrease of 3 for t_n. Thus, $t_n = -3n + b$.

Use the ordered pair $(1, -5)$ to find b.

$$-5 = -3(1) + b$$
$$-2 = b$$

Therefore, $t_n = -3n - 2$.

$$t_5 = -3(5) - 2 \qquad t_6 = -3(6) - 2$$
$$= -17 \qquad\qquad = -20$$

7. a.

n	1	2	3	4	5
t_n	2	5	10	17	26

Since an increase of 1 in n does not produce a constant increase in t_n, try the finite differences method.

n	t_n
1	2
2	5
3	10
4	17
5	26

3 5 7 9

2 2 2

Since the constant difference 2 occurs at the second level, the general term is of degree 2.

If $t_n = n^2$ is used, the ordered pairs $(1, 1)$, $(2, 4)$, $(3, 9)$, $(4, 16)$, and $(5, 25)$ are obtained. Note that the required values of t_n are 1 more than the second elements in the ordered pairs.

Therefore, the general term is $t_n = n^2 + 1$.

$$t_6 = 6^2 + 1 = 37 \qquad t_7 = 7^2 + 1 = 50$$

b.

n	1	2	3	4	5
t_n	−2	1	6	13	22

Since an increase of 1 in n does not produce a constant increase in t_n, try the finite differences method.

n	t_n	First differences	Second differences
1	−2		
		3	
2	1		2
		5	
3	6		2
		7	
4	13		2
		9	
5	22		

Since the constant difference 2 occurs at the second level, the general term must contain a perfect square.

If $t_n = n^2$ is used, the ordered pairs $(1,1)$, $(2,4)$, $(3,9)$, $(4,16)$, and $(5,25)$ are obtained.

The required values of t_n are three less than the second elements in those ordered pairs.

Therefore, the general term is $t_n = n^2 - 3$.

$$t_6 = 6^2 - 3 = 33 \qquad t_7 = 7^2 - 3 = 46$$

8. The sequence is $-10, -6, -2, 2, 6, \ldots$

n	1	2	3	4	5
t_n	−10	−6	−2	2	6

From the table of values, an increase of 1 in n produces an increase of 4 in t_n. Thus, $t_n = 4n + b$.

Use the ordered pair $(4, 2)$ to determine b.

$$2 = 4(4) + b$$
$$-14 = b$$

$$\therefore t_n = 4n - 14$$
$$t_{15} = 4(15) - 14 = 60 - 14 = 46$$

9. The sequence is 36, 31, 26, 21, 16,...

n	1	2	3	4	5
t_n	36	31	26	21	16

From the table of values, an increase of 1 in n produces a decrease of 5 in t_n. Thus, $t_n = -5n + b$.

Use the ordered pair $(1, 36)$ to determine b.

$$\begin{aligned} 36 &= -5(1) + b \\ 41 &= b \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= -5n + 41 \\ t_{11} &= -5(11) + 41 \\ &= -55 + 41 \\ &= -14 \end{aligned}$$

Section 1: Activity 3

1. a.

$$\begin{aligned} t_2 &= t_{2-1} + 7 \\ &= t_1 + 7 \\ &= 3 + 7 \\ &= 10 \end{aligned}$$

$$\begin{aligned} t_3 &= t_{3-1} + 7 \\ &= t_2 + 7 \\ &= 10 + 7 \\ &= 17 \end{aligned}$$

$$\begin{aligned} t_4 &= t_{4-1} + 7 \\ &= t_3 + 7 \\ &= 17 + 7 \\ &= 24 \end{aligned}$$

$$\begin{aligned} t_5 &= t_{5-1} + 7 \\ &= t_4 + 7 \\ &= 24 + 7 \\ &= 31 \end{aligned}$$

The first five terms are 3, 10, 17, 24, and 31.

b. To obtain t_2, let $n = 1$.

$$\begin{aligned} t_{1+1} &= t_1 - 6 \\ t_2 &= t_1 - 6 \\ &= -4 - 6 \\ &= -10 \end{aligned}$$

To obtain t_3, let $n = 2$.

$$\begin{aligned} t_{2+1} &= t_2 - 6 \\ t_3 &= t_2 - 6 \\ &= -10 - 6 \\ &= -16 \end{aligned}$$

To obtain t_4, let $n = 3$.

$$\begin{aligned} t_{3+1} &= t_3 - 6 \\ t_4 &= t_3 - 6 \\ &= -16 - 6 \\ &= -22 \end{aligned}$$

To obtain t_5, let $n = 4$.

$$\begin{aligned} t_{4+1} &= t_4 - 6 \\ t_5 &= t_4 - 6 \\ &= -22 - 6 \\ &= -28 \end{aligned}$$

The first five terms are -4, -10, -16, -22, and -28.

c. $$\begin{aligned} t_2 &= t_{2-1} + 8 \\ &= t_1 + 8 \\ &= -2 + 8 \\ &= 6 \end{aligned}$$

$$\begin{aligned} t_3 &= t_{3-1} + 8 \\ &= t_2 + 8 \\ &= 6 + 8 \\ &= 14 \end{aligned}$$

$$\begin{aligned} t_4 &= t_{4-1} + 8 \\ &= t_3 + 8 \\ &= 14 + 8 \\ &= 22 \end{aligned}$$

$$\begin{aligned} t_5 &= t_{5-1} + 8 \\ &= t_4 + 8 \\ &= 22 + 8 \\ &= 30 \end{aligned}$$

The first five terms are -2, 6, 14, 22, and 30.

d. To obtain t_2, let $n = 1$.

$$\begin{aligned} t_{1+1} &= t_1 - 10 \\ t_2 &= t_1 - 10 \\ &= 5 - 10 \\ &= -5 \end{aligned}$$

To obtain t_3, let $n = 2$.

$$\begin{aligned} t_{2+1} &= t_2 - 10 \\ t_3 &= t_2 - 10 \\ &= -5 - 10 \\ &= -15 \end{aligned}$$

To obtain t_4, let $n = 3$.

$$\begin{aligned} t_{3+1} &= t_3 - 10 \\ t_4 &= t_3 - 10 \\ &= -15 - 10 \\ &= -25 \end{aligned}$$

To obtain t_5, let $n = 4$.

$$\begin{aligned} t_{4+1} &= t_4 - 10 \\ t_5 &= t_4 - 10 \\ &= -25 - 10 \\ &= -35 \end{aligned}$$

The first five terms are 5, -5, -15, -25, and -35.

2. To obtain t_3, let $n = 2$.

$$\begin{aligned} t_{2+1} &= t_2 + t_{2-1} \\ t_3 &= t_2 + t_1 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

To obtain t_4, let $n = 3$.

$$\begin{aligned} t_{3+1} &= t_3 + t_{3-1} \\ t_4 &= t_3 + t_2 \\ &= 5 + 3 \\ &= 8 \end{aligned}$$

To obtain t_5, let $n = 4$.

$$\begin{aligned} t_{4+1} &= t_4 + t_{4-1} \\ t_5 &= t_4 + t_3 \\ &= 8 + 5 \\ &= 13 \end{aligned}$$

The first five terms are 2, 3, 5, 8, and 13.

3. The first five terms of the Fibonacci sequence are 1, 1, 2, 3, and 5. Each term in the sequence is the sum of the two terms just before it.

$$\begin{aligned} t_6 &= t_5 + t_4 \\ &= 5+3 \\ &= 8 \end{aligned} \qquad \begin{aligned} t_7 &= t_6 + t_5 \\ &= 8+5 \\ &= 13 \end{aligned} \qquad \begin{aligned} t_8 &= t_7 + t_6 \\ &= 13+8 \\ &= 21 \end{aligned}$$

$$\begin{aligned} t_9 &= t_8 + t_7 \\ &= 21+13 \\ &= 34 \end{aligned} \qquad \begin{aligned} t_{10} &= t_9 + t_8 \\ &= 34+21 \\ &= 55 \end{aligned} \qquad \begin{aligned} t_{11} &= t_{10} + t_9 \\ &= 55+34 \\ &= 89 \end{aligned}$$

The eleventh term has a value of 89.

4. Each term in the sequence is obtained by subtracting 4 from the previous term.

$$\begin{aligned} t_1 &= 3 \\ t_2 &= -1 \quad \text{or} \quad t_2 = t_1 - 4 \\ t_3 &= -5 \quad \text{or} \quad t_3 = t_2 - 4 \end{aligned}$$

The *n*th term t_n is four less than the $(n-1)$ term. Therefore, the recursion formula is $t_n = t_{n-1} - 4$, where $n > 1$ and $t_1 = 3$ or $t_{n+1} = t_n - 4$, where $t_1 = 3$.

5. Each term in the sequence is obtained by multiplying the previous term by -4.

$$\begin{aligned} t_1 &= 4 \\ t_2 &= -16 \quad \text{or} \quad t_2 = -4t_1 \\ t_3 &= 64 \quad \text{or} \quad t_3 = -4t_2 \end{aligned}$$

The *n*th term t_n is (-4) times the $(n-1)$ term. Therefore, the recursion formula is $t_n = -4t_{n-1}$, where $n > 1$ and $t_1 = 4$ or $t_{n+1} = -4t_n$, where $t_1 = 4$.

Section 1: Activity 4

1. **a.** nonarithmetic **b.** arithmetic **c.** nonarithmetic
   d. arithmetic **e.** arithmetic **f.** arithmetic
   g. arithmetic **h.** nonarithmetic **i.** arithmetic

2. **a.** $5-1=4$

   b. $-11-(-8)=-11+8$
   $=-3$

   c. $18-22=-4$

   d. $-4-(-6)=2$

   e. $-1-(-6)=5$

   f. $2-7=-5$

   g. $38-30=8$

   h. $-5-(-20)=15$

   i. $(1+\sqrt{2})-1=\sqrt{2}$

   j. $(2+2i)-(3+4i)=2+2i-3-4i$
   $=-1-2i$

3. a. $2, 7, 12, 17, \ldots$
   b. $-3, 1, 5, 9, \ldots$
   c. $20, 17, 14, 11, \ldots$
   d. $-16, -10, -4, 2, \ldots$
   e. $-20, -22, -24, -26, \ldots$
   f. $2\sqrt{3}, 5\sqrt{3}, 8\sqrt{3}, 11\sqrt{3}, \ldots$

4. a. $u-4=d$ and $16-u=d$

$$\begin{aligned} \therefore\ u-4 &= 16-u \\ 2u &= 20 \\ u &= 10 \end{aligned}$$

Thus, $d=10-4=6$.

$$\begin{aligned} t &= 16+6 \\ &= 22 \end{aligned}$$

   b. $u-(-20)=d$ and $-36-u=d$

$$\begin{aligned} \therefore\ u+20 &= -36-u \\ 2u &= -56 \\ u &= -28 \end{aligned}$$

Thus, $d=-36-(-28)=-8$.

$$\begin{aligned} t &= -36+(-8) \\ &= -44 \end{aligned}$$

   c. $t-23=d$ and $-11-t=d$

$$\begin{aligned} \therefore\ t-23 &= -11-t \\ 2t &= 12 \\ t &= 6 \end{aligned}$$

Thus, $d=6-23=-17$.

$$\begin{aligned} u+(-17) &= 23 \\ u &= 40 \end{aligned}$$

5. a. $m=12$, $r=-8$, $t=-18$, $u=-28$
   b. $m=-8$, $r=-5$, $t=1$, $u=4$
   c. $m=4$, $r=-1$, $t=-6$, $u=-11$

Section 1: Activity 5

1. a. $a=40$ and $d=32-40$
$\qquad\qquad\qquad = -8$

$$\begin{aligned} t_n &= a+(n-1)d \\ t_{10} &= 40+(10-1)(-8) \\ &= 40-72 \\ &= -32 \end{aligned}$$

b. $a = -20$ and $d = -13 - (-20)$
$= 7$

$$\begin{aligned} t_n &= a + (n-1)d \\ t_{16} &= -20 + (16-1)(7) \\ &= -20 + 105 \\ &= 85 \end{aligned}$$

2. a. $t_n = 63$, $a = -15$, and $d = -9 - (-15)$
$= 6$

$$\begin{aligned} t_n &= a + (n-1)d \\ 63 &= -15 + (n-1)6 \\ 63 &= -15 + 6n - 6 \\ 84 &= 6n \\ n &= 14 \end{aligned}$$

The 14th term has a value of 63.

b. $t_n = -112$, $a = 8$, and $d = 3 - 8$
$= -5$

$$\begin{aligned} t_n &= a + (n-1)d \\ -112 &= 8 + (n-1)(-5) \\ -112 &= 8 - 5n + 5 \\ 5n &= 125 \\ n &= 25 \end{aligned}$$

There are 25 terms in the sequence.

3. a. When four arithmetic means are inserted between -8 and 7, there will be $2 + 4 = 6$ terms. Thus, substitute the values $t_n = 7$, $a = -8$, and $n = 6$ into $t_n = a + (n-1)d$ to determine the value of d.

$$\begin{aligned} t_n &= a + (n-1)d \\ 7 &= -8 + (6-1)d \\ 15 &= 5d \\ d &= 3 \end{aligned}$$

Thus, the arithmetic sequence is $-8, -5, -2, 1, 4, 7$. The four arithmetic means between -8 and 7 are -5, -2, 1, and 4.

b. There will be $5 + 2 = 7$ terms in the sequence. Substitute the values $t_n = -19$, $a = 23$, and $n = 7$ into $t_n = a + (n-1)d$.

$$\begin{aligned} t_n &= a + (n-1)d \\ -19 &= 23 + (7-1)d \\ -42 &= 6d \\ d &= -7 \end{aligned}$$

The arithmetic sequence is $23, 16, 9, 2, -5, -12, -19$. The five arithmetic means between 23 and -19 are 16, 9, 2, -5, and -12.

4. The common difference must be the same between consecutive terms.

$$d = t_2 - t_1 = (2+x)-(x+5) = -3$$

$$d = t_3 - t_2 = (2x+8)-(2+x) = x+6$$

$$\therefore\ -3 = x+6$$
$$x = -9$$

The first term is $x+5=-9+5=-4$.
The second term is $2+x=2+(-9)=-7$.
The third term is $2x+8=2(-9)+8=-10$.

5. a. Substitute $t_5=-9$ and $n=5$ into $t_n=a+(n-1)d$.

$$a+(5-1)d=-9$$
$$a+4d=-9 \quad (1)$$

Substitute $t_{20}=36$ and $n=20$ into $t_n=a+(n-1)d$.

$$a+(20-1)d=36$$
$$a+19d=36 \quad (2)$$

Subtract (1) from (2).

$$\begin{array}{r} a+19d=36 \\ \underline{a+\ \ 4d=-9} \\ 15d=45 \\ d=3 \end{array}$$

Substitute $d=3$ in (1).

$$a+4(3)=-9$$
$$a=-21$$

$$t_n=-21+(n-1)(3)$$
$$=3n-24$$

b. Substitute $t_8=-13$ and $n=8$ into $t_n=a+(n-1)d$.

$$a+(8-1)d=-13$$
$$a+7d=-13 \quad (1)$$

Substitute $t_{17}=-49$ and $n=17$ into $t_n=a+(n-1)d$.

$$a+(17-1)d=-49$$
$$a+16d=-49 \quad (2)$$

Subtract (1) from (2).

$$\begin{array}{r} a+16d=-49 \\ \underline{a+\ \ 7d=-13} \\ 9d=-36 \\ d=-4 \end{array}$$

Substitute $d = -4$ in 1.

$$\begin{aligned} a + 7(-4) &= -13 \\ a &= 15 \end{aligned}$$

$$\begin{aligned} t_n &= 15 + (n-1)(-4) \\ &= -4n + 19 \end{aligned}$$

6. Substitute $t_{15} = 68$ and $n = 15$ into $t_n = a + (n-1)d$.

$$\begin{aligned} a + (15-1)d &= 68 \\ a + 14d &= 68 \quad (1) \end{aligned}$$

Substitute $t_{30} = 158$ and $n = 30$ into $t_n = a + (n-1)d$.

$$\begin{aligned} a + (30-1)d &= 158 \\ a + 29d &= 158 \quad (2) \end{aligned}$$

Subtract 1 from 2.

$$\begin{array}{r} a + 29d = 158 \\ \underline{a + 14d = 68} \\ 15d = 90 \\ d = 6 \end{array}$$

Substitute $d = 6$ in 1.

$$\begin{aligned} a + 14(6) &= 68 \\ a &= -16 \end{aligned}$$

$$\begin{aligned} t_n &= -16 + (n-1)(6) \\ &= 6n - 22 \end{aligned}$$

$$\begin{aligned} t_{21} &= 6(21) - 22 \\ &= 104 \end{aligned}$$

7. The first multiple is $55(5 \times 11 = 55)$ and the last multiple is $300(5 \times 60 = 300)$.
The arithmetic sequence is $55, 60, 65, \ldots, 300$.

$a = 55$, $d = 5$, and $t_n = 300$

$$\begin{aligned} t_n &= a + (n-1)d \\ 300 &= 55 + (n-1)5 \\ 300 &= 55 + 5n - 5 \\ 250 &= 5n \\ n &= 50 \end{aligned}$$

There are 50 multiples of 5 between 52 and 303.

8. **a.** The simple interest on \$800 at 10% per year is $800(0.10) = \$80$.
The money owing at the end of the first year is \$880.
The money owing at the end of the second year is \$960.
The money owing at the end of the third year is \$1040.
The money owing at the end of the fourth year is \$1120.

The arithmetic sequence is $880, 960, 1040, 1120, \ldots$

$a = 880$ and $d = 80$

$$\begin{aligned} t_n &= 880 + (n-1)80 \\ &= 800 + 80n \end{aligned}$$

b. The simple interest on \$3000 at 11% per year is $3000(0.11) = \$330$.
The money owing at the end of the first year is \$3330.

$a = 3330$, $d = 330$, and $n = 8$

$$\begin{aligned} t_n &= a + (n-1)d \\ &= 3330 + (8-1)330 \\ &= 3330 + 2310 \\ &= 5640 \end{aligned}$$

Thus, \$5640 is required to pay off the loan at the end of eight years.

9. The employee's salary at the end of the first year is $17\,000 + 800 = 17\,800$.
The employee's salary at the end of the second year is $17\,800 + 800 = 18\,600$.
The arithmetic sequence is $17\,800, 18\,600, 19\,400, \ldots$

$a = 17\,800$ and $d = 800$

$$\begin{aligned} t_n &= a + (n-1)d \\ t_8 &= 17\,800 + (8-1)(800) \\ &= 17\,800 + 5600 \\ &= 23\,400 \end{aligned}$$

The employee has a salary of \$23 400 at the end of eight years.

10. **a.** After the first year, the car is worth $\$16\,800 - 4200 = \$12\,600$.
After the second year, the car is worth $\$12\,600 - 900 = \$11\,700$.

$\therefore\ a = 12\,600$ and $d = -900$

$$\begin{aligned} t_n &= a + (n-1)d \\ t_5 &= 12\,600 + (5-1)(-900) \\ &= 12\,600 - 3600 \\ &= 9000 \end{aligned}$$

The car is worth \$9000 after five years of use.

b.

$$\begin{aligned} t_n &= a+(n-1)d \\ 3600 &= 12\,600+(n-1)(-900) \\ 3600 &= 12\,600-900n+900 \\ 900n &= 9900 \\ n &= 11 \end{aligned}$$

The car is worth \$3600 after eleven years.

11. The arithmetic sequence is $300, 280, 260, \ldots$

$a = 300$ and $d = -20$

$$\begin{aligned} t_n &= a+(n-1)d \\ t_9 &= 300+(9-1)(-20) \\ &= 300-160 \\ &= 140 \end{aligned}$$

The jogger runs 140 m in the ninth minute.

12. The arithmetic sequence is $4.9, 14.7, 24.5, \ldots$

$a = 4.9$, $d = 9.8$, and $n = 12$

a.

$$\begin{aligned} t_n &= a+(n-1)d \\ t_{12} &= 4.9+(12-1)9.8 \\ &= 4.9+(11)(9.8) \\ &= 4.9+107.8 \\ &= 112.7 \text{ m} \end{aligned}$$

The object falls 112.7 m in the 12th second.

b. The height of the helicopter above the water is $4.9+14.7+24.5+34.3+44.1+53.9+63.7+73.5+83.3+93.1+102.9+112.7 = 705.6$ m.

The helicopter is 705.6 m above the surface of the water.

Section 1: Activity 6

1. **a.** nongeometric

b. nongeometric

c. geometric
The common ratio is -3.

d. geometric
The common ratio is $\frac{1}{3}$.

e. geometric
The common ratio is $-\frac{1}{2}$.

f. nongeometric

g. geometric
The common ratio is $\frac{1}{2}$.

h. geometric
The common ratio is -2.

2. a. $-5 \div (-10) = \frac{1}{2}$

b. $5 \div (-1) = -5$

c. $5 \div 2 = \frac{5}{2}$

d. $-15 \div 45 = -\frac{1}{3}$

e. $-12 \div 3 = -4$

f. $-20 \div 80 = -\frac{1}{4}$

g. $\frac{4}{5} \div 4 = \frac{1}{5}$

h. $200 \div 300 = \frac{2}{3}$

3. a. $5,\ -10,\ 20,\ -40, \ldots$

b. $-2,\ \frac{2}{3},\ -\frac{2}{9},\ \frac{2}{27}, \ldots$

c. $-10,\ 30,\ -90,\ 270, \ldots$

d. $-60,\ 90,\ -135,\ \frac{405}{2}, \ldots$

e. $5,\ 1,\ \frac{1}{5},\ \frac{1}{25}, \ldots$

f. $3,\ 3\sqrt{7},\ 21,\ 21\sqrt{7}, \ldots$

4. a. $m = -\frac{2}{7}$

$s = 2$

$t = 98$

b. $m = -9$

$s = -36$

$t = -72$

c. $m = 81$

$s = -27$

$t = -3$

d. $m = 64$

$s = 4$

$t = 1$

Section 1: Activity 7

1. a. When $n = 1$, $t_1 = 2\left(\frac{1}{4}\right)^{1-1}$

$= 2(1)$

$= 2$

When $n = 2$, $t_2 = 2\left(\frac{1}{4}\right)^{2-1}$

$= 2\left(\frac{1}{4}\right)$

$= \frac{1}{2}$

When $n = 3$, $t_3 = 2\left(\frac{1}{4}\right)^{3-1}$

$= 2\left(\frac{1}{4}\right)^2$

$= \frac{1}{8}$

The first three terms are 2, $\frac{1}{2}$, and $\frac{1}{8}$.

b. When $n = 1$, $t_1 = 3\left(-\frac{2}{3}\right)^1$

$= -2$

When $n = 2$, $t_2 = 3\left(-\frac{2}{3}\right)^2$

$= 3\left(\frac{4}{9}\right)$

$= \frac{4}{3}$

When $n=3$, $$t_3 = 3\left(-\frac{2}{3}\right)^3 = 3\left(-\frac{8}{27}\right) = -\frac{8}{9}$$

The first three terms are -2, $\frac{4}{3}$, and $-\frac{8}{9}$.

2. **a.** $a=-7$ and $r=-3$

$$\begin{aligned} t_n &= ar^{n-1} \\ t_5 &= (-7)(-3)^{5-1} \\ &= (-7)(-3)^4 \\ &= (-7)(81) \\ &= -567 \end{aligned}$$

The fifth term is -567. The general term is $t_n = (-7)(-3)^{n-1}$.

b. $$a = \frac{1}{16} \text{ and } r = \left(-\frac{1}{8}\right) \div \frac{1}{16} = \left(-\frac{1}{8}\right) \times \frac{16}{1} = -2$$

$$\begin{aligned} t_n &= ar^{n-1} \\ t_5 &= \left(\frac{1}{16}\right)(-2)^{5-1} \\ &= \left(\frac{1}{16}\right)(-2)^4 \\ &= \frac{1}{16}(16) \\ &= 1 \end{aligned}$$

The fifth term is 1. The general term is $t_n = \left(\frac{1}{16}\right)(-2)^{n-1}$.

3. **a.** $t_n = 6144$, $a=-3$, and $r=-2$

$$\begin{aligned} t_n &= ar^{n-1} \\ 6144 &= (-3)(-2)^{n-1} \\ -2048 &= (-2)^{n-1} \\ (-2)^{11} &= (-2)^{n-1} \\ 11 &= n-1 \\ n &= 12 \end{aligned}$$

There are 12 terms in the sequence.

b. $t_n = -\frac{512}{66}$, $a = \frac{1}{66}$, and $r = \left(-\frac{1}{33}\right) \div \frac{1}{66}$

$$= \left(-\frac{1}{33}\right) \times \left(\frac{66}{1}\right)$$
$$= -2$$

$$t_n = ar^{n-1}$$
$$-\frac{512}{66} = \left(\frac{1}{66}\right)(-2)^{n-1}$$
$$-512 = (-2)^{n-1}$$
$$(-2)^9 = (-2)^{n-1}$$
$$9 = n-1$$
$$n = 10$$

There are 10 terms in the sequence.

4. **a.** $t_n = ar^{n-1}$

$$t_5 = ar^{5-1}$$
$$768 = ar^4$$

$$t_n = ar^{n-1}$$
$$t_7 = ar^{7-1}$$
$$12\,288 = ar^6$$

Divide ar^6 by ar^4.

$$\frac{ar^6}{ar^4} = \frac{12\,288}{768}$$
$$r^2 = 16$$
$$r = \pm 4$$

If $r = 4$ and $t_5 = 768$,

then $768 = ar^4$

$$768 = a(4)^4$$
$$768 = 256a$$
$$a = 3$$

$$t_n = ar^{n-1}$$
$$= 3(4)^{n-1}$$

If $r = 4$, the first three terms are 3, 12, and 48.

If $r = -4$ and $t_5 = 768$,

then $768 = ar^4$

$$768 = a(-4)^4$$
$$768 = 256a$$
$$a = 3$$

$$t_n = ar^{n-1}$$
$$= 3(-4)^{n-1}$$

If $r = -4$, the first three terms are 3, -12, and 48.

b. $t_n = ar^{n-1}$

$$t_3 = ar^{3-1}$$
$$7 = ar^2$$

$$t_n = ar^{n-1}$$
$$t_6 = ar^{6-1}$$
$$\frac{7}{8} = ar^5$$

Divide ar^5 by ar^2.

$$\frac{ar^5}{ar^2} = \frac{\frac{7}{8}}{7}$$

$$r^3 = \frac{7}{8} \times \frac{1}{7} = \frac{1}{8}$$

$$r = \frac{1}{2}$$

$r = \frac{1}{2}$ and $t_3 = 7$

$$7 = ar^2$$

$$7 = a\left(\frac{1}{2}\right)^2$$

$$7 = a\left(\frac{1}{4}\right)$$

$$a = 28$$

$$t_n = ar^{n-1}$$

$$= 28\left(\frac{1}{2}\right)^{n-1}$$

The first three terms are 28, 14, and 7.

5. **a.** There will be $3+2=5$ terms in the geometric sequence.

$t_n = -80$, $a = -5$, and $n = 5$

$$t_n = ar^{n-1}$$

$$-80 = (-5)(r)^4$$

$$16 = r^4$$

$$r = \pm 2$$

If $r = 2$, the sequence is $-5,\ -10,\ -20,\ -40,\ -80$. The geometric means are -10, -20, and -40.

If $r = -2$, the sequence is $-5,\ 10,\ -20,\ 40,\ -80$. The geometric means are 10, -20, and 40.

b. There will be $3+2=5$ terms in the geometric sequence.

$t_n = 324$, $a = 4$, and $n = 5$

$$t_n = ar^{n-1}$$

$$324 = 4(r)^4$$

$$81 = r^4$$

$$r = \pm 3$$

If $r = 3$, the sequence is $4,\ 12,\ 36,\ 108,\ 324$. The geometric means are 12, 36, and 108.

If $r=-3$, the sequence is $4, -12, 36, -108, 324$. The geometric means are -12, 36, and -108.

6. The common ratio is constant for a geometric sequence.

$$\begin{aligned}\frac{t_2}{t_1} &= \frac{t_3}{t_2}\\ \frac{x-10}{2x+4} &= \frac{x-1}{x-10}\\ (x-10)(x-10) &= (2x+4)(x-1)\\ x^2-20x+100 &= 2x^2+2x-4\\ 0 &= x^2+22x-104\\ 0 &= (x-4)(x+26)\end{aligned}$$

$(x-4)=0$ or $x+26=0$

$x=4 \qquad x=-26$

If $x=4$, then the three terms are as follows:

$$\begin{aligned}2x+4 &= 2(4)+4\\ &=12\end{aligned} \qquad \begin{aligned}x-10 &= 4-10\\ &=-6\end{aligned} \qquad \begin{aligned}x-1 &= 4-1\\ &=3\end{aligned}$$

For $x=4$, then the common ratio is $-\frac{1}{2}$.

If $x=-26$, then the three terms are as follows:

$$\begin{aligned}2x+4 &= 2(-26)+4\\ &=-48\end{aligned} \qquad \begin{aligned}x-10 &= -26-10\\ &=-36\end{aligned} \qquad \begin{aligned}x-1 &= -26-1\\ &=-27\end{aligned}$$

For $x=-26$, the common ratio is $\frac{3}{4}$.

7. a. $$\begin{aligned}i &= 11\%\\ &= \frac{11}{100}\\ &= 0.11\end{aligned} \qquad n=6$$

b. $$\begin{aligned}i &= \frac{9\%}{2}\\ &= 4.5\%\\ &= \frac{4.5}{100}\\ &= 0.045\end{aligned} \qquad \begin{aligned}n &= 2\times 7\\ &= 14\end{aligned}$$

c. $$\begin{aligned}i &= \frac{12.5\%}{4}\\ &= 3.125\%\\ &= \frac{3.125}{100}\\ &= 0.031\,25\end{aligned} \qquad \begin{aligned}n &= 4\times 5\\ &= 20\end{aligned}$$

d. $i = \frac{10\%}{12}$ $\qquad n = 12 \times 3$

$$\begin{aligned} i &= \frac{10\%}{12} & n &= 12 \times 3 \\ &\doteq 0.83\% & &= 36 \\ &\doteq \frac{0.83}{100} \\ &\doteq 0.0083 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 1600(1+0.0525)^{18} \\ &= 1600(1.0525)^{18} \\ &= \$4019.00 \end{aligned}$$

8. a. $P = 800$, $n = 7$, and $i = 11\%$

$$\begin{aligned} i &= 11\% \\ &= \frac{11}{100} \\ &= 0.11 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 800(1+0.11)^7 \\ &= 800(1.11)^7 \\ &= \$1660.93 \end{aligned}$$

b. $P = 1600$, $n = 2 \times 9$, and $i = \frac{10.5\%}{2}$

$$\begin{aligned} n &= 2 \times 9 & i &= \frac{10.5\%}{2} \\ &= 18 & &= 5.25\% \\ & & &= \frac{5.25}{100} \\ & & &= 0.0525 \end{aligned}$$

c. $P = 3400$, $n = 4 \times 6$, and $i = \frac{9.5\%}{4}$

$$\begin{aligned} n &= 4 \times 6 & i &= \frac{9.5\%}{4} \\ &= 24 & &= 2.375\% \\ & & &= \frac{2.375}{100} \\ & & &= 0.023\ 75 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 3400(1+0.023\ 75)^{24} \\ &= 3400(1.023\ 75)^{24} \\ &= \$5972.18 \end{aligned}$$

d. $P = 2200$, $n = 12 \times 8$, and $i = \frac{9\%}{12}$

$$\begin{aligned} n &= 12 \times 8 & i &= \frac{9\%}{12} \\ &= 96 & &= 0.75\% \\ & & &= \frac{0.75}{100} \\ & & &= 0.0075 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 2200(1+0.0075)^{96} \\ &= 2200(1.0075)^{96} \\ &= \$4507.63 \end{aligned}$$

e. $P = 1500,\ n = 6 \times 5 = 30$, and $i = \dfrac{12\%}{6} = 2\% = \dfrac{2}{100} = 0.02$

$$\begin{aligned} A &= P(1+i)^n \\ &= 1500(1+0.02)^{30} \\ &= 1500(1.02)^{30} \\ &= \$2717.04 \end{aligned}$$

9. a. $$\begin{aligned} A &= P(1+i)^n \\ &= 2100(1+0.09)^{6} \\ &= 2100(1.09)^{6} \\ &= \$3521.91 \end{aligned}$$

b. $P = 2100,\ n = 2 \times 6 = 12$, and $i = \dfrac{9\%}{2} = 4.5\% = \dfrac{4.5}{100} = 0.045$

$$\begin{aligned} A &= P(1+i)^n \\ &= 2100(1+0.045)^{12} \\ &= 2100(1.045)^{12} \\ &= \$3561.35 \end{aligned}$$

c. $P = 2100,\ n = 4 \times 6 = 24$, and $i = \dfrac{9\%}{4} = 2.25\% = \dfrac{2.25}{100} = 0.0225$

$$\begin{aligned} A &= P(1+i)^n \\ &= 2100(1+0.0225)^{24} \\ &= 2100(1.0225)^{24} \\ &= \$3582.11 \end{aligned}$$

d. $P = 2100$, $n = 12 \times 6$, and $i = \dfrac{9\%}{12}$

$$\begin{aligned} n &= 72 \\ i &= 0.75\% \\ &= \frac{0.75}{100} \\ &= 0.0075 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 2100(1+0.0075)^{72} \\ &= 2100(1.0075)^{72} \\ &= \$3596.36 \end{aligned}$$

10. $P = 2000$, $n = 2 \times 17$, and $i = \dfrac{11\%}{2}$

$$\begin{aligned} n &= 34 \\ i &= 5.5\% \\ &= \frac{5.5}{100} \\ &= 0.055 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 2000(1+0.055)^{34} \\ &= 2000(1.055)^{34} \\ &= \$12\,348.48 \end{aligned}$$

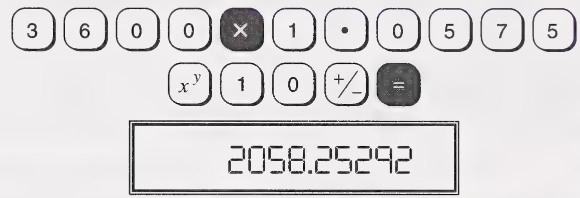
There will be $12 348.48 saved when the child turns seventeen.

11. $A = 3600$, $n = 2 \times 5$, and $i = \dfrac{11.5\%}{2}$

$$\begin{aligned} n &= 10 \\ i &= 5.75\% \\ &= \frac{5.75}{100} \\ &= 0.0575 \end{aligned}$$

$$\begin{aligned} P_v &= A(1+i)^{-n} \\ &= 3600(1+0.0575)^{-10} \\ &= 3600(1.0575)^{-10} \end{aligned}$$

Use a calculator to find P_v.



Tamika would need to invest $2058.25 today.

12. The sequence is 1, 4, 16, 64,...

$a = 1$ and $r = 4$

$$t_n = ar^{n-1}$$
$$t_8 = 1(4)^{8-1}$$
$$= 1(4)^7$$
$$= 16\,384$$

By the eighth hour, there are 16 384 bacteria.

13. **Method 1**

$t_5 = 3240$ and $r = 3$,

and $n = \frac{20}{5} + 1$
$= 5$

$$t_n = ar^{n-1}$$
$$3240 = a(3)^{5-1}$$
$$3240 = a(3)^4$$
$$3240 = a(81)$$
$$a = \frac{3240}{81}$$
$$= 40$$

Method 2

years	0	5	10	15	20
n	1	2	3	4	5
t_n	x	$3x$	$9x$	$27x$	$81x$

$$81x = 3240$$
$$x = 40$$

Mr. Jacobs began with 40 coins.

14. The population after one year is $P = 3600(1.04)^1$.

The population after four years is $P = 3600(1.04)^4$
$\doteq 4211.490\,816$

The expected population after four years is about 4211 zebras.

Section 1: Follow-up Activities

Extra Help

1. a. A **sequence** of numbers is a set of numbers arranged in a definite order.

b. A sequence is a function whose **domain** is the natural numbers and whose **range** is the set of real or complex numbers.

c. A sequence that has no last term is called an **infinite** sequence.

d. A sequence that has a last term is called a **finite** sequence.

e. When a sequence of numbers is given, then this sequence is usually the **range** of the sequence.

f. The **terms** in a sequence are separated by commas.

2. $t_1 = -(1)^2 + 3(1)$
$= -1 + 3$
$= 2$

$t_2 = -(2)^2 + 3(2)$
$= -4 + 6$
$= 2$

$t_3 = -(3)^2 + 3(3)$
$= -9 + 9$
$= 0$

$t_4 = -(4)^2 + 3(4)$
$= -16 + 12$
$= -4$

$t_5 = -(5)^2 + 3(5)$
$= -25 + 15$
$= -10$

The first five terms are 2, 2, 0, -4, and -10.

3. $t_1 = -4(1) + 9$
$= 5$

$t_2 = -4(2) + 9$
$= 1$

$t_3 = -4(3) + 9$
$= -3$

$t_4 = -4(4) + 9$
$= -7$

The first four terms are 5, 1, -3, and -7.

$t_{17} = -4(17) + 9$
$= -68 + 9$
$= -59$

The sequence is $5,\ 1,\ -3,\ -7, \ldots,\ -59$.

4.

n	1	2	3	4
t_n	7	10	15	22

Since an increase of 1 in n does not produce a constant increase in t_n, try the finite differences method.

n	t_n		
1	7		
		3	
2	10		2
		5	
3	15		2
		7	
4	22		

Since the constant difference 2 occurs at the second level, the general term must contain a perfect square. If $t_n = n^2$ is used, the ordered pairs are $(1,1)$, $(2,4)$, $(3,9)$, and $(4,16)$. The required values of t_n are six more than the second elements in the ordered pairs.

Therefore, the general term is $t_n = n^2 + 6$.

$t_5 = 5^2 + 6$
$= 31$

$t_6 = 6^2 + 6$
$= 42$

5. The sequence is $20, 14, 8, 2, -4, -10, \ldots$.

n	1	2	3	4
t_n	20	14	8	2

An increase of 1 in n produces a decrease of 6 in t_n. Thus, $t_n = -6n + b$.

Use the ordered pair $(1, 20)$ to determine b.

$$\begin{aligned} 20 &= -6(1) + b \\ 26 &= b \end{aligned}$$

Therefore, $t_n = -6n + 26$.

$$\begin{aligned} t_{16} &= -6(16) + 26 \\ &= -96 + 26 \\ &= -70 \end{aligned}$$

6. To obtain t_2, let $n = 1$.

$$\begin{aligned} t_{1+1} &= t_1 + 8 \\ t_2 &= t_1 + 8 \\ &= 4 + 8 \\ &= 12 \end{aligned}$$

To obtain t_3, let $n = 2$.

$$\begin{aligned} t_{2+1} &= t_2 + 8 \\ t_3 &= t_2 + 8 \\ &= 12 + 8 \\ &= 20 \end{aligned}$$

To obtain t_4, let $n = 3$.

$$\begin{aligned} t_{3+1} &= t_3 + 8 \\ t_4 &= t_3 + 8 \\ &= 20 + 8 \\ &= 28 \end{aligned}$$

The first four terms are 4, 12, 20, and 28.

7. Each term is six less than the previous term.

$$\begin{aligned} t_1 &= -4 \\ t_2 &= -10 \quad \text{or} \quad t_2 = t_1 - 6 \\ t_3 &= -16 \quad \text{or} \quad t_3 = t_2 - 6 \end{aligned}$$

The *n*th term t_n is 6 less than the $(n-1)$ term. Therefore, the recursion formula is $t_n = t_{n-1} - 6$, $n > 1$, where $t_1 = -4$ or $t_{n+1} = t_n - 6$, where $t_1 = -4$.

8. **a.** $a = -5$ and $d = -4$ **b.** $a = 19$ and $d = -3$
c. $a = -8$ and $d = 5$ **d.** $a = 16$ and $d = -10$

9. $t_n = -310$, $a = 20$, and $d = 5 - 20$
$$= -15$$

$$\begin{aligned} t_n &= a + (n-1)d \\ -310 &= 20 + (n-1)(-15) \\ -310 &= 20 - 15n + 15 \\ -345 &= -15n \\ n &= 23 \end{aligned}$$

The 23rd term is -310.

10. $a = -64$, $n = 45$, and $d = -56 - (-64)$
$$= 8$$

$$\begin{aligned} t_n &= a + (n-1)d \\ t_{45} &= -64 + (45-1)(8) \\ &= -64 + 352 \\ &= 288 \end{aligned}$$

The 45th term is 288.

11. $a = 56$, $n = 18$, and $d = 47 - 56$
$$= -9$$

$$\begin{aligned} t_n &= a + (n-1)d \\ t_{18} &= 56 + (18-1)(-9) \\ &= 56 - 153 \\ &= -97 \end{aligned}$$

$$\begin{aligned} t_n &= 56 + (n-1)(-9) \\ &= 56 - 9n + 9 \\ &= -9n + 65 \end{aligned}$$

12. There will be $4 + 2 = 6$ terms in the sequence.

$\therefore\ n = 6$, $a = -32$, and $t_n = 28$

$$\begin{aligned} t_n &= a + (n-1)d \\ 28 &= -32 + (6-1)d \\ 28 &= -32 + 5d \\ 60 &= 5d \\ d &= 12 \end{aligned}$$

The arithmetic sequence is -32, -20, -8, 4, 16, 28. The four arithmetic means are -20, -8, 4, and 16.

13. Substitute $t_3 = -12$ and $n = 3$ in the formula $t_n = a + (n-1)d$.

$$-12 = a + (3-1)d$$
$$-12 = a + 2d \quad (1)$$

Substitute $t_{14} = 54$ and $n = 14$ in the formula $t_n = a + (n-1)d$.

$$54 = a + (14-1)d$$
$$54 = a + 13d \quad (2)$$

Subtract (1) from (2).

$$\begin{aligned} 54 &= a + 13d \\ \underline{-12} &\underline{= a + 2d} \\ 66 &= 11d \\ d &= 6 \end{aligned}$$

The common difference is 6.

To obtain the first term, substitute $d = 6$ in (1).

$$\begin{aligned} -12 &= a + 2(6) \\ a &= -24 \end{aligned}$$

The first term is -24.

Substitute $d = 6$ and $a = -24$ into the formula.

$$\begin{aligned} t_n &= a + (n-1)d \\ &= -24 + (n-1)(6) \\ &= -24 + 6n - 6 \\ &= 6n - 30 \end{aligned}$$

The general term is $t_n = 6n - 30$.

14. The common difference must be the same between consecutive terms.

$$\begin{aligned} d &= t_2 - t_1 \\ &= (2x+5) - (2x-2) \\ &= 7 \end{aligned} \qquad \begin{aligned} d &= t_3 - t_2 \\ &= (x+5) - (2x+5) \\ &= -x \end{aligned}$$

Since $d = d$, then $7 = -x$; thus, $x = -7$.

first term: $\begin{aligned} 2x - 2 &= 2(-7) - 2 \\ &= -16 \end{aligned}$

second term: $\begin{aligned} 2x + 5 &= 2(-7) + 5 \\ &= -9 \end{aligned}$

third term: $\begin{aligned} x + 5 &= -7 + 5 \\ &= -2 \end{aligned}$

15. Let x be the middle term.

The sequence is $\frac{1}{2}$, x, 32.

$$\frac{x}{\frac{1}{2}} = \frac{32}{x}$$
$$(x)(x) = \left(\frac{1}{2}\right)(32)$$
$$x^2 = 16$$
$$x = \pm 4$$

The geometric mean is either 4 or -4.

16. Let m be the tenth term (t_{10}).

The sequence is $96,\ m,\ 1536$.

$$\frac{m}{96} = \frac{1536}{m}$$
$$(m)(m) = (96)(1536)$$
$$m^2 = 147\,456$$
$$m = \pm 384$$

$$\begin{aligned} r &= \frac{m}{96} \\ &= \frac{\pm 384}{96} \\ &= \pm 4 \end{aligned}$$

The common ratio is either 4 or -4.

17. Let x be the middle term.

The geometric sequence is $-\frac{1}{3},\ x,\ -243$.

$$\frac{x}{-\frac{1}{3}} = \frac{-243}{x}$$
$$x^2 = \left(-\frac{1}{3}\right)(-243)$$
$$x^2 = 81$$
$$x = \pm 9$$

The positive geometric mean is 9.

18. Let m represent the middle term.

The geometric sequence is $45,\ m,\ 180$.

$$\frac{m}{45} = \frac{180}{m}$$
$$(m)(m) = (45)(180)$$
$$m^2 = 8100$$
$$m = \pm 90$$

The remaining brother recieved $90.

Enrichment

1. Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	2	6	18	54

Each term of the sequence is obtained by multiplying the previous term by 3. Each term of the sequence cannot be expressed as a power of 3, but it may be possible to express each term as a multiple of a power of 3. Set up a table of values showing n, t_n, and 3^n.

n	1	2	3	4
t_n	2	6	18	54
3^n	3	9	27	81

Note that 3^n is used since the t_n values are increasing when n increases.

Try the general term $t_n = k(3^n)$.

When $n = 1$, $t_n = 2$.

$$\therefore\ 2 = k(3^1)$$
$$k = \frac{2}{3}$$

Thus, $t_n = \frac{2}{3}(3^n)$. Check if this formula satisfies the other values of n and t_n.

If $n = 4$, $t_4 = \frac{2}{3}(3^4)$
$$= \tfrac{2}{3}(81)$$
$$= 54$$

This agrees with the fourth term in the sequence. Thus, the general term is $t_n = \frac{2}{3}(3^n)$.

2. Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	136	68	34	17

Each term of the sequence is obtained by multiplying the previous term by $\frac{1}{2}$. Each term of the sequence cannot be expressed as a power of 2, but it may be possible to express each term as a multiple of a power of 2. Set up a table of values showing n, t_n, and 2^{-n}.

n	1	2	3	4
t_n	136	68	34	17
2^{-n}	2^{-1}	2^{-2}	2^{-3}	2^{-4}

Note that 2^{-n} is used since the t_n values are decreasing when n increases.

Try the general term $t_n = k(2^{-n})$. When $n=1$, $t_n = 136$.

$$\therefore\ 136 = k(2^{-1})$$
$$136 = k\left(\frac{1}{2^1}\right)$$
$$k = 272$$

Thus, $t_n = 272(2^{-n})$ or $272\left(\frac{1}{2^n}\right) = \frac{272}{2^n}$. Check if this formula satisfies the other values of n and t_n. Let $n=3$.

$$t_3 = \frac{272}{2^3}$$
$$= \frac{272}{8}$$
$$= 34$$

This agrees with the given values. Thus, the general term is $t_n = \frac{272}{2^n}$.

3. Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	5	10	20	40

Each term of the sequence is obtained by multiplying the previous term by 2. Each term of the sequence can probably be expressed as a multiple of a power of 2. Use a table of values showing n, t_n, and 2^n.

n	1	2	3	4
t_n	5	10	20	40
2^n	2	4	8	16

Try the general term $t_n = k(2^n)$. When $n=1$, $t_n = 5$.

$$\therefore\ 5 = k(2^1)$$
$$k = 2.5$$

Thus, $t_n = 2.5(2^n)$. Check if this formula satisfies the other values of n and t_n. Let $n=3$.

$$t_3 = 2.5(2^3)$$
$$= 2.5(8)$$
$$= 20$$

This agrees with the given values. Thus, the general term is $t_n = 2.5(2^n)$.

4. Set up a table of values showing n and t_n.

n	1	2	3	4
t_n	108	36	12	4

Each term is obtained by multiplying the previous term by $\frac{1}{3}$. Each term of the sequence can probably be expressed as a multiple of a power of 3. Use a table of values showing n, t_n, and 3^{-n}.

n	1	2	3	4
t_n	108	36	12	4
3^{-n}	3^{-1}	3^{-2}	3^{-3}	3^{-4}

Note that 3^{-n} is used since the t_n values are decreasing when n increases.

Try the general term $t_n = k\left(3^{-n}\right)$. When $n = 2$, $t_n = 36$.

$$\begin{aligned} \therefore\ 36 &= k\left(3^{-2}\right) \\ 36 &= k\left(\frac{1}{3^2}\right) \\ k &= 9(36) \\ &= 324 \end{aligned}$$

Thus, $t_n = 324\left(3^{-n}\right)$ or $\frac{324}{3^n}$.

Check if this formula satsifies the other values of n and t_n. Let $n = 3$.

$$\begin{aligned} t_3 &= \frac{324}{3^3} \\ &= \frac{324}{27} \\ &= 12 \end{aligned}$$

This agrees with the third term in the sequence. Thus, the general term is $t_n = \frac{324}{3^n}$.

5. The common difference d is $\frac{6}{y} - \frac{3}{y} = \frac{3}{y}$.

$$\begin{aligned} t_n &= a + (n-1)d \\ &= \frac{3}{y} + (n-1)\left(\frac{3}{y}\right) \\ &= \frac{3}{y} + \frac{3n}{y} - \frac{3}{y} \\ &= \frac{3n}{y} \end{aligned}$$

The general term is $t_n = \frac{3n}{y}$.

$$\begin{aligned} t_7 &= \frac{3(7)}{y} \\ &= \frac{21}{y} \end{aligned}$$

The seventh term is $\frac{21}{y}$.

6. The common difference d is $(5u-t)-(4u-3t)=u+2t$.

$$\begin{aligned} t_n &= a+(n-1)d \\ &= (4u-3t)+(n-1)(u+2t) \\ &= 4u-3t+nu+2nt-u-2t \\ &= 3u-5t+nu+2nt \end{aligned}$$

Substitute $13u+15t$ for t_n.

$$\begin{aligned} 13u+15t &= 3u-5t+nu+2nt \\ 10u+20t &= n(u+2t) \\ \frac{10u+20t}{(u+2t)} &= n \\ n &= 10 \end{aligned}$$

7. Remember that the terms in an arithmetic sequence are $a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ldots$
Thus, the first term is a and the third term is $a+2d$.

$$\begin{aligned} a+(a+2d) &= 32 \\ 2a+2d &= 32 \quad (1) \end{aligned}$$

The second term is $a+d$ and the fourth term is $a+3d$.

$$\begin{aligned} (a+d)+(a+3d) &= 24 \\ 2a+4d &= 24 \quad (2) \end{aligned}$$

Subtract (1) from (2).

$$\begin{aligned} 2a+4d &= 24 \\ \underline{2a+2d} &\underline{= 32} \\ 2d &= -8 \\ d &= -4 \end{aligned}$$

Substitute $d=-4$ in (2).

$$\begin{aligned} 2a+4(-4) &= 24 \\ 2a &= 24+16 \\ 2a &= 40 \\ a &= 20 \end{aligned}$$

The sequence is $20,\ 16,\ 12,\ 8,\ldots$

8. The common difference is the same for an arithmetic sequence.

$$\begin{aligned} 10-r &= t-10 \\ 20 &= r+t \\ (r+t)^2 &= (20)^2 \\ r^2+2rt+t^2 &= 400 \\ 2rt &= 400-298 \qquad \left(\text{given } r^2+t^2=298\right) \\ 2rt &= 102 \\ rt &= 51 \\ r &= \frac{51}{t} \end{aligned}$$

$$r + t = 20$$

$$\frac{51}{t} + t = 20$$

$$51 + t^2 = 20t \qquad (\text{Multiply by } t.)$$

$$t^2 - 20t + 51 = 0$$

$$(t-3)(t-17) = 0$$

$$t - 3 = 0 \quad \text{or} \quad t - 17 = 0$$

$$t = 3 \qquad\qquad t = 17$$

Substitute $t = 3$ in $20 = r + t$.

$$20 = r + 3$$

$$r = 17$$

The sequence is $17,\ 10,\ 3, \ldots$

Substitute $t = 17$ in $20 = r + t$.

$$20 = r + 17$$

$$r = 3$$

The sequence is $3,\ 10,\ 17, \ldots$

Therefore, the arithmetic sequence is $17,\ 10,\ 3, \ldots$ or $3,\ 10,\ 17, \ldots$

9. $A = 475,\ P = 400$, and $i = \frac{12}{6}$

$$= 2\%$$

$$= \frac{2}{100}$$

$$= 0.02$$

The variable n represents the number of two-month intervals.

$$A = P(1+i)^n$$

$$475 = 400(1+0.02)^n$$

$$4.75 = 4(1.02)^n \qquad (\text{Divide by } 100.)$$

Solve for n by using logarithms.

$$\log 4.75 = \log\left[4(1.02)^n\right]$$

$$\log 4.75 = \log 4 + n \log 1.02$$

$$\log 4.75 - \log 4 = n \log 1.02$$

$$n = \frac{\log 4.75 - \log 4}{\log 1.02}$$

Use a calculator to find n.

(4 • 7 5 log − 4 log) ÷ 1 • 0 2 log =

8.678 154386

The number of interest periods is 9 (rounded to the next higher period). Thus, Sam must wait 9 two-month intervals or $\frac{9}{6} = 1.5$ years to purchase the microwave.

10. $t_3 = ar^2$ and $t_4 = ar^3$

$$t_3 + t_4 = \frac{3}{4}$$
$$ar^2 + ar^3 = \frac{3}{4}$$
$$ar^2(1+r) = \frac{3}{4} \quad (1)$$

$t_6 = ar^5$ and $t_7 = ar^6$

$$t_6 + t_7 = -6$$
$$ar^5 + ar^6 = -6$$
$$ar^5(1+r) = -6 \quad (2)$$

Divide equation (2) by (1).

$$\frac{ar^5(1+r)}{ar^2(1+r)} = \frac{-6}{\frac{3}{4}}$$
$$r^3 = (-6)\left(\frac{4}{3}\right)$$
$$r^3 = -8$$
$$r = -2$$

Substitute $r = -2$ in (1).

$$a(-2)^2[1+(-2)] = \frac{3}{4}$$
$$a(4)(-1) = \frac{3}{4}$$
$$-4a = \frac{3}{4}$$
$$a = -\frac{3}{16}$$

The first seven terms of the sequence are $-\frac{3}{16}, \frac{3}{8}, -\frac{3}{4}, \frac{3}{2}, -3,$ 6, and -12.

Section 2: Activity 1

1. infinite

2. finite

3. infinite

4. finite

Section 2: Activity 2

1. $$\begin{aligned} S_1 &= 2(1)^2 + 1 \\ &= 3 \end{aligned} \qquad t_1 = 3$$

$$\begin{aligned} S_2 &= 2(2)^2 + 2 \\ &= 10 \end{aligned} \qquad \begin{aligned} t_2 &= S_2 - S_1 \\ &= 10 - 3 \\ &= 7 \end{aligned}$$

$$\begin{aligned} S_3 &= 2(3)^2 + 3 \\ &= 21 \end{aligned} \qquad \begin{aligned} t_3 &= S_3 - S_2 \\ &= 21 - 10 \\ &= 11 \end{aligned}$$

$$\begin{aligned} S_4 &= 2(4)^2 + 4 \\ &= 36 \end{aligned} \qquad \begin{aligned} t_4 &= S_4 - S_3 \\ &= 36 - 21 \\ &= 15 \end{aligned}$$

The first four terms are 3, 7, 11, and 15.

2. **a.** $$\begin{aligned} t_n &= S_n - S_{n-1} \\ &= \left(n^2 + n\right) - \left[(n-1)^2 + (n-1)\right] \\ &= \left(n^2 + n\right) - \left[n^2 - 2n + 1 + n - 1\right] \\ &= \cancel{n^2} + \cancel{n} - \cancel{n^2} + 2n - \cancel{1} - \cancel{n} + \cancel{1} \\ &= 2n \end{aligned}$$

b. $$\begin{aligned} t_n &= 2n \\ t_6 &= 2(6) \\ &= 12 \end{aligned}$$

c. $$\begin{aligned} S_n &= n^2 + n \\ S_6 &= (6)^2 + 6 \\ &= 42 \end{aligned}$$

Section 2: Activity 3

1. **a.** arithmetic series
   b. not an arithmetic series (It is a geometric series.)
   c. not an arithmetic series
   d. arithmetic series
   e. not an arithmetic series (It is an arithmetic sequence.)

2. $$\begin{aligned} S &= 5 + 9 + 13 + 17 + 21 + 25 + 29 + 33 \\ S &= 33 + 29 + 25 + 21 + 17 + 13 + 9 + 5 \\ \hline 2S &= 38 + 38 + 38 + 38 + 38 + 38 + 38 + 38 \\ 2S &= 8(38) \\ S &= \frac{8(38)}{2} \\ &= 4(38) \\ &= 152 \end{aligned}$$

3. $S_n = \frac{n}{2}(a + t_n)$

$S_8 = \frac{8}{2}(5+33)$

$= 4(38)$

$= 152$

c. $S_n = \frac{n}{2}(a + t_n)$

$S_{11} = \frac{11}{2}[21 + (-9)]$

$= \frac{11}{2}[12]$

$= 66$

d. $S_n = \frac{n}{2}(a + t_n)$

$S_{32} = \frac{32}{2}[-40 + (-195)]$

$= 16[-235]$

$= -3760$

4. $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_8 = \frac{8}{2}[2(5) + (8-1)4]$

$= 4[10 + (7)4]$

$= 4(38)$

$= 152$

Section 2: Activity 4

1. a. $S_n = \frac{n}{2}(a + t_n)$

$S_{14} = \frac{14}{2}(-9+56)$

$= 7(47)$

$= 329$

b. $S_n = \frac{n}{2}(a + t_n)$

$S_{16} = \frac{16}{2}[12 + (-108)]$

$= 8[-96]$

$= -768$

2. a. $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{15} = \frac{15}{2}[2(1) + (15-1)(3)]$

$= \frac{15}{2}[2 + (14)(3)]$

$= \frac{15}{2}[44]$

$= 330$

b. $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_9 = \frac{9}{2}[2(-22) + (9-1)(4)]$

$= \frac{9}{2}[-44 + 32]$

$= \frac{9}{2}[-12]$

$= -54$

c. $$\begin{aligned} S_n &= \frac{n}{2}[2a+(n-1)d] \\ S_{18} &= \frac{18}{2}[2(19)+(18-1)(-3)] \\ &= 9[38+17(-3)] \\ &= 9[-13] \\ &= -117 \end{aligned}$$

d. $$\begin{aligned} S_n &= \frac{n}{2}[2a+(n-1)d] \\ S_{12} &= \frac{12}{2}[2(-14)+(12-1)(-3)] \\ &= 6[-28+11(-3)] \\ &= 6[-61] \\ &= -366 \end{aligned}$$

3. The terms in the arithmetic series which represent the multiples of 8 between 20 and 205 are $24+32+40+\ldots+200$.

$t_n = 200$, $a = 24$, and $d = 8$

The value of n must be determined so that the sum formula can be used.

$$\begin{aligned} t_n &= a+(n-1)d \\ 200 &= 24+(n-1)8 \\ 200 &= 24+8n-8 \\ 200 &= 16+8n \\ 184 &= 8n \\ n &= 23 \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}(a+t_n) \\ S_{23} &= \frac{23}{2}(24+200) \\ &= \frac{23}{2}(224) \\ &= 2576 \end{aligned}$$

The sum of the multiples of 8 between 20 and 205 is 2576.

4. $t_n = \frac{43}{3}$, $a = \frac{1}{3}$, and $d = \frac{6}{3}$
$= 2$

The value of n must be obtained before the sum formula can be used.

$$\begin{aligned} t_n &= a+(n-1)d \\ \frac{43}{3} &= \frac{1}{3}+(n-1)(2) \\ \frac{43}{3} &= \frac{1}{3}+2n-2 \\ \frac{48}{3} &= 2n \\ n &= \frac{48}{6} \\ &= 8 \end{aligned}$$

$$S_n = \frac{n}{2}(a+t_n)$$

$$S_8 = \frac{8}{2}\left(\frac{1}{3}+\frac{43}{3}\right)$$

$$= 4\left(\frac{44}{3}\right)$$

$$= 58\frac{2}{3}$$

5. $S_{14} = -28$ and $n = 14$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$-28 = \frac{14}{2}[2a+(14-1)d]$$

$-28 = 7[2a+13d]$ (Divide both sides by 7.)

$-4 = 2a+13d$ ①

$S_{26} = 572$ and $n = 26$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_{26} = \frac{26}{2}[2a+(26-1)d]$$

$572 = 13[2a+25d]$ (Divide both sides by 13.)

$44 = 2a+25d$ ②

Subtract ① from ②.

$$\begin{aligned} 44 &= 2a+25d \\ \underline{-4} &\underline{= 2a+13d} \\ 48 &= 12d \\ d &= 4 \end{aligned}$$

Substitute $d = 4$ into ①.

$$\begin{aligned} -4 &= 2a+13(4) \\ -56 &= 2a \\ a &= -28 \end{aligned}$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_{50} = \frac{50}{2}[2(-28)+(50-1)4]$$

$$= 25[-56+196]$$

$$= 25[140]$$

$$= 3500$$

6. **a.** $S_n = 210$, $a = 1$, and $d = 1$

$$
\begin{aligned}
S_n &= \tfrac{n}{2}[2a + (n-1)d] \\
210 &= \tfrac{n}{2}[2(1) + (n-1)1] \\
210 &= \tfrac{n}{2}[2 + n - 1] \\
210 &= \tfrac{n}{2}[1 + n] \\
420 &= n[1 + n] \\
420 &= n + n^2
\end{aligned}
$$

$$n^2 + n - 420 = 0$$
$$(n+21)(n-20) = 0$$
$$n + 21 = 0 \quad \text{or} \quad n - 20 = 0$$
$$n = -21 \qquad\qquad n = 20$$

Therefore, $n = 20$ since -21 is inadmissible.

b. $S_n = 90$, $a = -20$, and $d = 5$

$$
\begin{aligned}
S_n &= \tfrac{n}{2}[2a + (n-1)d] \\
90 &= \tfrac{n}{2}[2(-20) + (n-1)(5)] \\
90 &= \tfrac{n}{2}[-40 + 5n - 5] \\
90 &= \tfrac{n}{2}[-45 + 5n] \\
180 &= n[-45 + 5n] \\
180 &= -45n + 5n^2
\end{aligned}
$$

$$5n^2 - 45n - 180 = 0$$
$$5\left(n^2 - 9n - 36\right) = 0$$
$$5(n+3)(n-12) = 0$$
$$n + 3 = 0 \quad \text{or} \quad n - 12 = 0$$
$$n = -3 \qquad\qquad n = 12$$

Therefore, $n = 12$ since -3 is inadmissible.

c. $S_n = -184$, $a = 11$, and $d = -3$

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a+(n-1)d] \\
-184 &= \frac{n}{2}[2(11)+(n-1)(-3)] \\
-184 &= \frac{n}{2}[22-3n+3] \\
-184 &= \frac{n}{2}[25-3n] \\
-368 &= 25n-3n^2
\end{aligned}
$$

$$
\begin{aligned}
3n^2-25n-368 &= 0 \\
(3n+23)(n-16) &= 0
\end{aligned}
$$

$3n+23=0$ or $n-16=0$

$n=-\frac{23}{3}$ $\quad$ $n=16$

Therefore, $n=16$ since $-\frac{23}{3}$ is inadmissible.

7. The arithmetic series is $1+3+5+\ldots$

$S_n = 289$, $a=1$, and $d=2$

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a+(n-1)d] \\
289 &= \frac{n}{2}[2(1)+(n-1)2] \\
289 &= \frac{n}{2}[2+2n-2] \\
289 &= \frac{n}{2}[2n] \\
n^2 &= 289 \\
n &= 17
\end{aligned}
$$

There must be 17 consecutive, odd natural numbers.

8. $160+240+320+\ldots$

$a=160$, $d=80$, and $n=8$

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a+(n-1)d] \\
S_8 &= \frac{8}{2}[2(160)+(8-1)80] \\
&= 4[320+7(80)] \\
&= 4[320+560] \\
&= 4[880] \\
&= 3520
\end{aligned}
$$

In eight years, the repairs will cost \$3520.

9. The series is $26+25+24+\ldots+10$.

$t_n=10$, $a=26$, and $d=-1$

The variable n must be determined before the sum formula is used.

$$
\begin{aligned}
t_n &= a+(n-1)d \\
10 &= 26+(n-1)(-1) \\
10 &= 26-n+1 \\
n &= 27-10 \\
&= 17
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a+(n-1)d] \\
S_{17} &= \frac{17}{2}[2(26)+(17-1)(-1)] \\
&= \frac{17}{2}[52+(-16)] \\
&= \frac{17}{2}(36) \\
&= 306
\end{aligned}
$$

The total number of posts is 306.

10. The series is $0.35+0.40+0.45+\ldots$

$S_n=32.25$, $a=0.35$, and $d=0.05$

$$
\begin{aligned}
S_n &= \frac{n}{2}[2(a)+(n-1)d] \\
32.25 &= \frac{n}{2}[2(0.35)+(n-1)(0.05)] \\
32.25 &= \frac{n}{2}[0.70+0.05n-0.05] \\
32.25 &= \frac{n}{2}[0.65+0.05n] \\
64.50 &= 0.65n+0.05n^2
\end{aligned}
$$

$$
\begin{aligned}
0.05n^2+0.65n-64.50 &= 0 \\
5n^2+65n-6450 &= 0 \\
5\left(n^2+13n-1290\right) &= 0 \\
5(n-30)(n+43) &= 0
\end{aligned}
$$

$n-30=0 \quad \text{or} \quad n+43=0$

$n=30 \qquad n=-43$ (inadmissible)

The child saves for 30 weeks.

11. The depreciation the first year is $\$5500\times0.12=\$660$. The depreciation the second year is $\$660+\$70=\$730$ and so on. The arithmetic series for the depreciation is $660+730+800+\ldots$

$a=660$, $d=70$, and $n=5$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_5 = \frac{5}{2}[2(660)+(5-1)70]$$

$$= \frac{5}{2}[1600]$$

$$= 4000$$

The depreciation at the end of five years is \$4000. Therefore, the value of the motorcycle is $\$5500 - \$4000 = \$1500$.

12. **a.** To find the distance travelled in the tenth second, the sequence 4.9, 14.7, 24.5,… must be used.

$a = 4.9$, $n = 10$, and $d = 9.8$ (gravity)

$$t_n = a+(n-1)d$$

$$t_{10} = 4.9+(10-1)(9.8)$$

$$= 4.9+(9)(9.8)$$

$$= 4.9+88.2$$

$$= 93.1$$

The object falls 93.1 m in the tenth second.

b. To find the total distance fallen in ten seconds, the series $4.9+14.7+24.5+\ldots$ must be used.

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_{10} = \frac{10}{2}[2(4.9)+(10-1)(9.8)]$$

$$= 5[9.8+9(9.8)]$$

$$= 5[9.8+88.2]$$

$$= 5[98]$$

$$= 490 \text{ m}$$

In ten seconds, the object falls 490 m. Therefore, the helicopter is 490 m above the ground surface.

Section 2: Activity 5

1. **a.** not a geometric series (It is an arithmetic series.)
b. geometric series
c. geometric series
d. not a geometric series (It is an arithmetic series.)
e. not a geometric series (It is a geometric sequence.)

2. $a = -3$ and $r = 4$

Multiply S by the common ratio 4.

$$4S = -12+(-48)+\ldots+(-3072)+(-12\,288)$$

Now write the expressions so that the $4S$ is the top expression and the S is the bottom expression. Subtract S from $4S$.

$$\begin{aligned}
4S &= \phantom{(-3)+{}} -12+(-48)+\ldots+(-3072)+(-12\,288) \\
S &= (-3)+-12+(-48)+\ldots+(-3072) \\
\hline
3S &= 3 \qquad\qquad\qquad\qquad -12\,288 \\
3S &= -12\,285 \\
S &= -4095
\end{aligned}$$

3. First, n must be determined.

$$\begin{aligned}
t_n &= ar^{n-1} \\
-3072 &= -3(4)^{n-1} \\
1024 &= 4^{n-1} \\
4^5 &= 4^{n-1} \\
5 &= n-1 \\
6 &= n
\end{aligned}$$

$$\begin{aligned}
S_n &= \frac{a\left(r^n-1\right)}{r-1} \\
S_6 &= \frac{-3\left(4^6-1\right)}{4-1} \\
&= \frac{-3(4096-1)}{3} \\
&= -1(4095) \\
&= -4095
\end{aligned}$$

This sum agrees with the sum obtained in question 1.

Section 2: Activity 6

1. a. $a=18$, $n=9$, and $r=\frac{6}{18}$

$$= \frac{1}{3}$$

Since $|r|<1$, $S_n = \dfrac{a\left(1-r^n\right)}{1-r}$

$$\begin{aligned}
S_9 &= \frac{18\left[1-\left(\frac{1}{3}\right)^9\right]}{1-\frac{1}{3}} \\
&= \frac{18\left[1-\frac{1}{19\,683}\right]}{\frac{2}{3}} \\
&= \frac{3}{2}(18)\left[\frac{19\,682}{19\,683}\right] \\
&= 27\left[\frac{19\,682}{19\,683}\right] \\
&= \frac{19\,682}{729} \\
&= 26\frac{728}{729}
\end{aligned}$$

b. $a = -4$, $n = 8$, and $r = \dfrac{8}{-4}$

$$= -2$$

Since $|r| > 1$, $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$\begin{aligned} S_8 &= \frac{-4\left[(-2)^8 - 1\right]}{-2-1} \\ &= \frac{-4[256-1]}{-3} \\ &= \frac{4}{3}[255] \\ &= 340 \end{aligned}$$

c. $a = -2$, $n = 10$, and $r = \dfrac{-6}{-2}$

$$= 3$$

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_{10} &= \frac{(-2)\left[3^{10} - 1\right]}{3-1} \\ &= \frac{(-2)(59\,049 - 1)}{2} \\ &= -59\,048 \end{aligned}$$

d. $a = 100$, $n = 5$, and $r = \dfrac{-20}{100}$

$$= -\frac{1}{5}$$

Since $|r| < 1$, $S_n = \dfrac{a(1 - r^n)}{1 - r}$

$$\begin{aligned} S_5 &= \frac{100\left[1 - \left(-\frac{1}{5}\right)^5\right]}{1 - \left(-\frac{1}{5}\right)} \\ &= \frac{100\left[1 - \left(-\frac{1}{3125}\right)\right]}{\frac{6}{5}} \\ &= \frac{5}{6}(100)\left(\frac{3126}{3125}\right) \\ &= 83.36 \end{aligned}$$

2. a. $t_n = 1944$, $a = -8$, and $r = \dfrac{24}{-8}$

$$= -3$$

The value of n must be determined.

$$\begin{aligned} t_n &= ar^{n-1} \\ 1944 &= (-8)(-3)^{n-1} \\ -243 &= (-3)^{n-1} \\ (-3)^5 &= (-3)^{n-1} \\ 5 &= n-1 \\ 6 &= n \end{aligned}$$

Since $|r| > 1$, use the following formula:

$$\begin{aligned} S_n &= \frac{a\left(r^n - 1\right)}{r-1} \\ S_6 &= \frac{(-8)\left[(-3)^6 - 1\right]}{-3-1} \\ &= \frac{-8[729-1]}{-4} \\ &= 2[728] \\ &= 1456 \end{aligned}$$

b. $t_n = -\frac{9}{8}$, $a = 36$, and $r = \frac{-18}{36}$

$$= -\frac{1}{2}$$

The value of n must be determined.

$$\begin{aligned} t_n &= ar^{n-1} \\ -\frac{9}{8} &= (36)\left(-\frac{1}{2}\right)^{n-1} \\ -\frac{9}{8} \times \frac{1}{36} &= \left(-\frac{1}{2}\right)^{n-1} \\ -\frac{1}{32} &= \left(-\frac{1}{2}\right)^{n-1} \\ \left(-\frac{1}{2}\right)^5 &= \left(-\frac{1}{2}\right)^{n-1} \\ 5 &= n-1 \\ 6 &= n \end{aligned}$$

Since $|r| < 1$, use the following formula:

$$\begin{aligned} S_n &= \frac{a\left(1 - r^n\right)}{1-r} \\ S_6 &= \frac{36\left[1 - \left(-\frac{1}{2}\right)^6\right]}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{36\left[1 - \frac{1}{64}\right]}{\frac{3}{2}} \\ &= \frac{2}{3}(36)\left(\frac{63}{64}\right) \\ &= \frac{189}{8} \text{ or } 23\frac{5}{8} \end{aligned}$$

3. $S_n = 2188,\ a = 4$, and $r = \dfrac{-12}{4}$

$$= -3$$

$$\begin{aligned}
\text{Since } |r|>1,\ S_n &= \frac{a\left(r^n-1\right)}{r-1}\\
2188 &= \frac{4\left[(-3)^n-1\right]}{-3-1}\\
2188 &= \frac{4\left[(-3)^n-1\right]}{-4}\\
-2188 &= (-3)^n-1\\
-2187 &= (-3)^n\\
(-3)^7 &= (-3)^n\\
n &= 7
\end{aligned}$$

The first seven terms produce a sum of 2188.

4. $S_5 = -11\frac{5}{8},\ n = 5$, and $r = \dfrac{1}{2}$

$$\begin{aligned}
\text{Since } |r|<1,\ S_n &= \frac{a\left(1-r^n\right)}{1-r}\\
-11\frac{5}{8} &= \frac{a\left[1-\left(\frac{1}{2}\right)^5\right]}{1-\frac{1}{2}}\\
-\frac{93}{8} &= \frac{a\left[1-\frac{1}{32}\right]}{\frac{1}{2}}\\
-\frac{93}{8} &= \frac{a\left[\frac{31}{32}\right]}{\frac{1}{2}}\\
-\frac{93}{8} &= a\left(\frac{31}{\cancel{32}_{16}}\right)\left(\frac{\cancel{2}^{1}}{1}\right)\\
a &= \left(-\frac{\cancel{93}^{3}}{\cancel{8}_{1}}\right)\left(\frac{\cancel{16}^{2}}{\cancel{31}_{1}}\right)\\
&= -6
\end{aligned}$$

To obtain $1a$, multiply both sides by $\frac{16}{31}$.

The first term is -6.

5. $t_7 = 243$, $n = 7$, and $r = 3$

$$\begin{aligned} t_n &= ar^{n-1} \\ 243 &= a(3)^{7-1} \\ 3^5 &= a(3^6) \\ a &= \frac{3^5}{3^6} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r-1} \\ S_8 &= \frac{\frac{1}{3}[(3)^8 - 1]}{3-1} \\ &= \frac{\frac{1}{3}(6561-1)}{2} \\ &= \frac{1}{2}\left(\frac{1}{3}\right)(6560) \\ &= 1093\frac{1}{3} \end{aligned}$$

6. **a.** $a = 400\,000$, $n = 5$, and $r = \frac{2}{5}$
$= 0.4$

$$\begin{aligned} t_n &= ar^{n-1} \\ t_5 &= 400\,000(0.4)^{5-1} \\ &= 400\,000(0.4)^4 \\ &= 400\,000(0.0256) \\ &= 10\,240 \end{aligned}$$

The fifth prize is worth $\$10\,240$.

b.
$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1-r} \\ S_7 &= \frac{400\,000(1 - 0.4^7)}{1 - 0.4} \\ &= \frac{400\,000(1 - 0.001\,6384)}{0.6} \\ &= \frac{400\,000(0.998\,3616)}{0.6} \\ &= 665\,574.40 \end{aligned}$$

The total amount of money for the first seven prizes is $\$665\,574.40$.

7. There were six payments made at the end of the third year.
$a = 4000$, $n = 6$, and $r = 0.95$

$$\begin{aligned} S_n &= \frac{a\left(1-r^n\right)}{1-r} \\ S_6 &= \frac{4000(1-0.95)^6}{1-0.95} \\ &= \frac{4000(1-0.735\,091\,89)}{0.05} \\ &= 21\,192.65 \end{aligned}$$

The pensioner will receive \$21 192.65.

8. The series is $10 + 6 + 3.6 + \ldots$

$a = 10$, $n = 4$, and $r = 0.6$

$$\begin{aligned} S_n &= \frac{a\left(1-r^n\right)}{1-r} \\ S_4 &= \frac{10\left(1-0.6^4\right)}{1-0.6} \\ &= \frac{10(1-0.1296)}{0.4} \\ &= 25(0.8704) \\ &= 21.76 \end{aligned}$$

The total distance after 4 h is 21.76 km.

Section 2: Activity 7

1. a. $$\begin{aligned} \sum_{n=2}^{6} 4-n &= (4-2)+(4-3)+(4-4)+(4-5)+(4-6) \\ &= 2+1+0+(-1)+(-2) \end{aligned}$$

b. $$\begin{aligned} \sum_{k=2}^{5} 3^{k^2-1} &= \left(3^{2^2-1}\right)+\left(3^{3^2-1}\right)+\left(3^{4^2-1}\right)+\left(3^{5^2-1}\right) \\ &= 3^3 + 3^8 + 3^{15} + 3^{24} \end{aligned}$$

c. $$\begin{aligned} \sum_{n=4}^{7} (n-2)^2 &= \left[(4-2)^2\right]+\left[(5-2)^2\right]+\left[(6-2)^2\right] \\ &\quad +\left[(7-2)^2\right] \\ &= 2^2+3^2+4^2+5^2 \\ &= 4+9+16+25 \end{aligned}$$

d. $$\begin{aligned} \sum_{n=3}^{6} 2^n &= 2^3+2^4+2^5+2^6 \\ &= 8+16+32+64 \end{aligned}$$

2. a. $S=[2(1)-3]+[2(2)-3]+[2(3)-3]+[2(4)-3]$

$\quad+[2(5)-3]+[2(6)-3]$

$=[2-3]+[4-3]+[6-3]+[8-3]+[10-3]+[12-3]$

$=-1+1+3+5+7+9$

$=24$

b. Note that the sum required will be for the terms t_3, t_4, t_5, t_6, and t_7.

$S=[5(3)+2]+[5(4)+2]+[5(5)+2]+[5(6)+2]$

$\quad+[5(7)+2]$

$=[15+2]+[20+2]+[25+2]+[30+2]+[35+2]$

$=17+22+27+32+37$

$=135$

c. $S=3^1+3^2+3^3+3^4+3^5$

$=3+9+27+81+243$

$=363$

d. The sum required will be for the terms t_2, t_3, t_4, t_5, and t_6.

$S=(2-4)^2+(3-4)^2+(4-4)^2+(5-4)^2+(6-4)^2$

$=4+1+0+1+4$

$=10$

e. $S=\left[-4^1+(-4)^1\right]+\left[-4^2+(-4)^2\right]+\left[-4^3+(-4)^3\right]$

$\quad+\left[-4^4+(-4)^4\right]$

$=\left[-4+(-4)\right]+\left[-16+16\right]+\left[-64+(-64)\right]$

$\quad+\left[-256+256\right]$

$=-8+0-128+0$

$=-136$

3. a. This is an arithmetic series where $a=3$ and $d=2$.

$t_n=a+(n-1)d$

$=3+(n-1)2$

$=3+2n-2$

$=1+2n$

The number of the last term must be determined.

$25=1+2n$

$24=2n$

$n=12$

The summation notation is $\sum_{n=1}^{12} 1+2n$.

b. This is a geometric series where $a = 3$ and $r = 5$.

$$\begin{aligned} t_n &= ar^{n-1} \\ &= 3(5)^{n-1} \end{aligned}$$

The number of the last term must be determined.

$$\begin{aligned} 1875 &= 3(5)^{n-1} \\ 625 &= 5^{n-1} \\ 5^4 &= 5^{n-1} \\ 4 &= n-1 \quad \text{(Equate the exponents since the bases are equal.)} \\ n &= 5 \\ t_5 &= 1875 \end{aligned}$$

The summation notation is $\sum_{n=1}^{5} 3(5)^{n-1}$.

c. This is a geometric series where $r = 2$ and $a = \frac{1}{4}$.

$$\begin{aligned} t_n &= ar^{n-1} \\ &= \left(\frac{1}{4}\right)(2)^{n-1} \quad \text{or} \\ &= \left(\frac{1}{2^2}\right)(2)^{n-1} \\ &= \left(2^{-2}\right)\left(2^{n-1}\right) \\ &= 2^{-2+n-1} \\ &= 2^{n-3} \end{aligned}$$

The number of the last term must be determined.

$$\begin{aligned} 128 &= \left(\frac{1}{4}\right)(2)^{n-1} \\ 512 &= 2^{n-1} \\ 2^9 &= 2^{n-1} \\ 9 &= n-1 \\ n &= 10 \end{aligned}$$

The summation notation is $\sum_{n=1}^{10} 2^{n-3}$.

d. This is an arithmetic series where $a = -4$ and $d = 2$.

$$\begin{aligned} t_n &= a + (n-1)d \\ &= -4 + (n-1)(2) \\ &= -4 + 2n - 2 \\ &= 2n - 6 \end{aligned}$$

The number of the last term must be determined.

$$\begin{aligned} 18 &= 2n - 6 \\ 24 &= 2n \\ n &= 12 \end{aligned}$$

The summation notation is $\sum_{n=1}^{12} 2n-6$.

e. $3+12+27+48+75+108+147$

$$\begin{array}{cccccc} 9 & 15 & 21 & 27 & 33 & 39 \\ & 6 & 6 & 6 & 6 & 6 \end{array}$$

The second level differences are the same; thus, part of the general term must be n^2. Use one of the terms of the series to find the complete general term. Try the term 48.

$$48 = \_\_\_ n^2$$

$$48 = \_\_\_ 4^2 \quad (n = 4 \text{ for the fourth term which is 48.})$$

$$48 = 3(4)^2$$

Thus, the general term is $3n^2$ and the summation notation is

$$\sum_{n=1}^{7} 3n^2.$$

Section 2: Follow-up Activities

Extra Help

1. **a.** Substitute 6, 7, 8, 9, and 10 for n in $4n+5$.

$$\begin{aligned}\sum_{n=6}^{10} 4n+5 &= [4(6)+5]+[4(7)+5]+[4(8)+5] \\ &\quad +[4(9)+5]+[4(10)+5] \\ &= (24+5)+(28+5)+(32+5)+(36+5) \\ &\quad +(40+5) \\ &= 29+33+37+41+45\end{aligned}$$

b. Substitute 3, 4, 5, 6, and 7 for k in $\frac{2}{k+1}$.

$$\begin{aligned}\sum_{k=3}^{7} \frac{2}{k+1} &= \left(\frac{2}{3+1}\right)+\left(\frac{2}{4+1}\right)+\left(\frac{2}{5+1}\right)+\left(\frac{2}{6+1}\right) \\ &\quad +\left(\frac{2}{7+1}\right) \\ &= \frac{2}{4}+\frac{2}{5}+\frac{2}{6}+\frac{2}{7}+\frac{2}{8}\end{aligned}$$

2. **a.** Substitute 3, 4, and 5 for n in 2^{n+1}.

$$\begin{aligned}\sum_{n=3}^{5} 2^{n+1} &= 2^{3+1}+2^{4+1}+2^{5+1} \\ &= 2^4+2^5+2^6 \\ &= 16+32+64 \\ &= 112\end{aligned}$$

b. Substitute 7, 8, and 9 for k in k^2-6k.

$$\begin{aligned}\sum_{k=7}^{9} k^2-6k &= \left[7^2-6(7)\right]+\left[8^2-6(8)\right]+\left[9^2-6(9)\right]\\ &=[49-42]+[64-48]+[81-54]\\ &=7+16+27\\ &=50\end{aligned}$$

3. The series is $16+9+2-5-12-19-26$.

$$\begin{aligned}S_1&=16\end{aligned}\qquad\begin{aligned}S_2&=16+9\\&=25\end{aligned}$$

$$\begin{aligned}S_3&=16+9+2\\&=27\end{aligned}\qquad\begin{aligned}S_4&=16+9+2-5\\&=22\end{aligned}$$

$$\begin{aligned}S_5&=16+9+2-5-12\\&=10\end{aligned}\qquad\begin{aligned}S_6&=16+9+2-5-12-19\\&=-9\end{aligned}$$

The sequence is $16,\ 25,\ 27,\ 22,\ 10,\ -9$.

4. $S_n=-55$, $a=-28$, and $\begin{aligned}[t]d&=(-23)-(-28)\\&=5\end{aligned}$

$$\begin{aligned}S_n&=\frac{n}{2}[2a+(n-1)d]\\ -55&=\frac{n}{2}[2(-28)+(n-1)5]\\ -55&=\frac{n}{2}[-56+5n-5]\\ -55&=\frac{n}{2}[-61+5n]\\ -110&=-61n+5n^2\end{aligned}$$

$$\begin{aligned}5n^2-61n+110&=0\\ (5n-11)(n-10)&=0\end{aligned}$$

$$5n-11=0 \quad\text{or}\quad n-10=0$$
$$n=\frac{11}{5} \qquad\qquad n=10$$

Since $\frac{11}{5}$ is inadmissible, there are 10 terms in the series.

5. $a = 2$, $n = 6$, and $r = \frac{1}{2}$

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{2\left[\left(\frac{1}{2}\right)^6 - 1\right]}{\frac{1}{2} - 1} \\ &= \frac{2\left[\frac{1}{64} - 1\right]}{-\frac{1}{2}} \\ &= (2)\left(-\frac{63}{64}\right)(-2) \\ &= 3\frac{15}{16} \end{aligned}$$

6. $S_n = 360$, $a = 243$, and $r = \frac{81}{243}$
$= \frac{1}{3}$

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ 360 &= \frac{243\left[\left(\frac{1}{3}\right)^n - 1\right]}{\frac{1}{3} - 1} \\ 360 &= \frac{243\left[\left(\frac{1}{3}\right)^n - 1\right]}{-\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{3}\right)^n - 1 &= \left(\overset{40}{\cancel{360}}\right)\left(-\frac{2}{3}\right)\left(\frac{1}{\underset{27}{\cancel{243}}}\right) \\ \left(\frac{1}{3}\right)^n - 1 &= -\frac{80}{81} \\ \left(\frac{1}{3}\right)^n &= -\frac{80}{81} + 1 \\ \left(\frac{1}{3}\right)^n &= \frac{1}{81} \\ \left(\frac{1}{3}\right)^n &= \left(\frac{1}{3}\right)^4 \\ n &= 4 \end{aligned}$$

7. $t_7 = \frac{1}{8}$, $n = 7$, and $r = -\frac{1}{2}$

$$t_n = ar^{n-1}$$
$$\frac{1}{8} = a\left(-\frac{1}{2}\right)^{7-1}$$
$$\frac{1}{8} = a\left(\frac{1}{-2^6}\right)$$
$$a = \frac{1}{8}(-2)^6$$
$$= 8$$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$
$$S_7 = \frac{8\left[\left(-\frac{1}{2}\right)^7 - 1\right]}{-\frac{1}{2}-1}$$
$$= \frac{8\left[-\frac{1}{128}-1\right]}{-\frac{3}{2}}$$
$$= (8)\left(-\frac{129}{128}\right)\left(-\frac{2}{3}\right)$$
$$= \left(\frac{1}{\cancel{8}}\right)\left(\frac{\overset{43}{\cancel{129}}}{\underset{\underset{8}{\cancel{16}}}{\cancel{128}}}\right)\left(\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}}\right)$$
$$= \frac{43}{8} \text{ or } 5\frac{3}{8}$$

Enrichment

1. $$\sum_{n=1}^{4}[3n+2n] = [3(1)+2(1)]+[3(2)+2(2)]+[3(3)+2(3)]$$
$$+[3(4)+2(4)]$$
$$= 5+10+15+20$$
$$= 50$$

$$\sum_{n=1}^{4}3n = 3(1)+3(2)+3(3)+3(4)$$
$$= 3+6+9+12$$
$$= 30$$

$$\sum_{n=1}^{4}2n = 2(1)+2(2)+2(3)+2(4)$$
$$= 2+4+6+8$$
$$= 20$$

Since $50 = 30 + 20$, then $\sum_{n=1}^{4}[3n+2n] = \sum_{n=1}^{4}3n + \sum_{n=1}^{4}2n$.

2. Substitute 1, 2, 3, for k into $(-1)^{k+1}(2k)$.

$$\begin{aligned}\sum_{k=1}^{3}(-1)^{k+1}(2k) &= \left[(-1)^{1+1}(2)(1)\right]+\left[(-1)^{2+1}(2)(2)\right]\\ &\quad+\left[(-1)^{3+1}(2)(3)\right]\\ &=\left[(-1)^{2}(2)\right]+\left[(-1)^{3}(4)\right]+\left[(-1)^{4}(6)\right]\\ &=2+(-4)+6\\ &=4\end{aligned}$$

3. The series is $5+7+9+\ldots+25$.

This is an arithmetic series where $a=5$ and $d=2$.

$$\begin{aligned}t_n &= a+(n-1)d\\ &=5+(n-1)2\\ &=5+2n-2\\ &=3+2n\end{aligned}$$

The number of the last term must be determined.

$$\begin{aligned}25&=3+2n\\ 2n&=22\\ n&=11\end{aligned}$$

The summation notation is $\sum_{n=1}^{11}2n+3$.

$$\begin{aligned}\sum_{n=1}^{11}2n+3 &= 5+7+9+11+13+15+17+19+21+23+25\\ &=165\end{aligned}$$

Colin ran 165 km in total.

4. Make a the subject of the formula. Fill in the values for S_n, r, and n in each of the respective cells.

For cell A4, type: = B4 * (C4 – 1)/(C4 ^ D4 – 1)

The display is as shown.

	A	B	C	D
1		a = Sn*(r-1)/(r^n-1)		
2				
3	a	Sn	r	n
4	14	560	3	4
5				

5. Fill the cells: $B5=-2520$, $C5=2$, $D5=6$. Copy A4 and paste in A5. You get the following display.

	A	B	C	D
1		a = Sn*(r-1)/(r^n-1)		
2				
3	a	Sn	r	n
4	14	560	3	4
5	-40	-2520	2	6

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